

Yang–Mills measure and the master field on the sphere*

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March 31, 2017

Abstract

Here is a first version of a study of the Yang–Mills measure on the sphere with structure group of large dimension. We focus in this article on unitary structure groups. We show that the Wilson loops converge in probability towards a deterministic field, called master field. As a corollary, we obtain that the Brownian loop in unitary matrices converges in non-commutative distribution as the dimension goes to infinity. This is a first instance of a unitary free Brownian loop.

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*Research supported by EPSRC grant EP/I03372X/1.

1 Introduction

The Yang–Mills measure, associated to a (two-dimensional) surface Σ and to a compact Lie group G , is a probability measure on (generalized) connections of principal G -bundles over Σ . It was introduced in a series of works by Gross et al. [23], Fine [16], Driver [14], Witten [42, 43], Sengupta [39] and Lévy [30], as a mathematical version of Euclidean Yang–Mills field theory. In this paper, we will consider the Yang–Mills measure in the case where the surface Σ is fixed and the group G is a classical matrix group of high dimension. The interest of such a set-up from the viewpoint of random matrix theory was first raised in the mathematics literature by Singer [41], who made several conjectures, based on earlier work in physics [20, 21, 26, 27]. The high-dimensional limit of the Yang–Mills measure when Σ is the whole plane has since been studied by Xu [44], Sengupta [40], Lévy [32], Anshelevich & Sengupta [1], Dahlqvist [8] and others. As we shall see, the general problem is closely related to another, addressed by Biane [3], Lévy [31], Lévy & Maïda [33] and Collins, Dahlqvist & Kemp [7], which is to understand the high-dimensional limit of Brownian motion on the group as a non-commutative process.

We focus here on the case where the surface Σ is a sphere. This has received particular attention in the physics literature [4, 9, 22, 38], as it displays a phase transition of third order named after Douglas and Kazakov [11]. A corresponding mathematical analysis of the partition function was achieved by Boutet de Monvel & Shcherbina [5] and Lévy & Maïda [34]. The main result of the present work, Theorem 2.5, confirms a conjecture of Singer [41], showing that the non-commutative process associated to the Yang–Mills measure on the sphere for the unitary group $U(N)$ converges as $N \rightarrow \infty$ to a limit ‘master field’, which we can characterize fully. As a by-product of this result, we also prove convergence in non-commutative distribution of the Brownian loop in $U(N)$.

There is a system of relations, discovered by Makeenko and Migdal [36], indexed by families of embedded loops, between the expectations under the Yang–Mills measure of polynomials in the traces of the loop holonomies. These have now been proved for the whole plane by Lévy [32] and Dahlqvist [8] and for any compact surface by Driver et al. [12]. The Makeenko–Migdal equations provide a potential line of argument to prove convergence of the Yang–Mills measure as $N \rightarrow \infty$, that is, to show a suitable concentration estimate for the holonomy traces, and to pass to the limit in the equations, showing that the limit equations determine a unique limit object. In the whole plane case, moment estimates for unitary Brownian motion provide the needed concentration, and allows to augment the Makeenko–Migdal equations by a further equation, such that the whole system of equations then characterizes the limit field. So the programme has been completed in that case [8, 32]. However, as noted in [12], the concentration and characterization problems have remained open in general.

In this paper, we will establish two key points. First, using the Makeenko–Migdal equations, we show in Section 3 that the convergence of marginals to a

deterministic limit for simple loops forces the same to hold for any loop. Then, we show in Proposition 4.1 that expectations and covariances of traces of simple loops can be represented by functional of a discrete β -ensemble. In Proposition 3.6, we see that this representation allows to identify the limit as $N \rightarrow \infty$, following the work of Johansson [25] and Féral [15] on discrete β -ensembles. This amounts to a rigorous version of ideas explained by Boulatov [4] and Douglas & Kazakov [11]. Following the physics literature, the limit trace on loops based at an arbitrary point is called here the *master field on the sphere*. We call the non-commutative limit of the Brownian loop the *free unitary loop*.

An alternative line of argument to the second step, that we shall discuss elsewhere, would be to use the fact that the process of eigenvalues of the marginals of the Brownian bridge is known to have the same law as a Dyson Brownian motion on the circle, starting from 1 and conditioned to return to 1. Indeed, several scaling limits of this conditioned process have recently been understood by Liechty & Wang [35]. This link was first observed in the physics literature in Forrester et al. [17, 18]. Section 4 gives another way to obtain macroscopic results on the empirical distribution of this process.

This paper is organized as follows. Section 2 introduces the model and our results. Section 3 shows how the Makeenko-Migdal equations can be used to show the convergence of the master field in probability, knowing convergence and concentration for simple loops. Section 4 shows that this latter condition is fulfilled, thanks to a duality relation with a discrete β -ensemble. Section 5 gives properties for the master field, including a relation with the free Brownian Bridge in the subcritical regime. Section 6 gives a formula for the evaluation of the master field on a large class of loops. In the appendix A, we recall and adapt the arguments of regularity of the Yang-Mills measure and of the master field, that we use here.

Subject to certain modifications, to be explained in a future work, the argument explained here applies to other series of compact groups and also to the projective plane in place of the sphere.

2 Setting and statement of the main result

We review the notion of a Yang-Mills holonomy field over a compact Riemann surface. We then discuss its relation, in the case of the sphere, to the Brownian bridge in a Lie group. Next we introduce the master field on loops in the sphere. Finally we state our main result on convergence of Yang-Mills holonomy in $U(N)$ over the sphere to the master field, and note its corollary for the non-commutative limit of the Brownian bridge in $U(N)$.

2.1 Yang-Mills measure on a compact Riemann surface

We recall in this subsection the approach of Lévy [30] to the Yang-Mills measure. Let Σ be a compact Riemann surface and let G be a compact Lie group. Write T for the total area of Σ and write 1 for the unit element of G . We fix a bi-invariant Riemannian metric on G and we denote the associated heat kernel by $p = (p_t(g) : t \in (0, \infty), g \in G)$. Thus p is the unique smooth positive function on

$(0, \infty) \times G$ such that

$$\frac{\partial p}{\partial t} = \frac{1}{2} \Delta p$$

and, for all continuous functions f on G ,

$$\int_G f(g) p_t(g) dg \rightarrow f(1), \quad \text{as } t \rightarrow 0.$$

Here we have written Δ for the Laplace–Beltrami operator and dg for the normalized Haar measure on G .

We specialize in later sections to the case where Σ is the sphere \mathbb{S}_T of area T , and where G is the group $U(N)$ of unitary $N \times N$ matrices. Then we use the following inner product on its Lie algebra $\mathfrak{u}(N)$, the space of skew-Hermitian matrices, to define the metric

$$\langle g_1, g_2 \rangle = N \text{Tr}(g_1 g_2^*) \quad (1)$$

where $\text{Tr}(g) = \sum_{i=1}^N g_{ii}$. This dependence of the metric on N , which is standard in random matrix theory, is chosen so that the objects of interest to us have a non-trivial scaling limit as $N \rightarrow \infty$.

Write $P(\Sigma)$ for the set of oriented paths of finite length in Σ , considered modulo reparametrization. We denote the length of a path $\gamma \in P(\Sigma)$ by $\mathcal{L}(\gamma)$. Each path γ has a starting point $\underline{\gamma}$ and an endpoint $\bar{\gamma}$. We write γ^{-1} for the reversion of γ , that is, the path of reverse orientation from $\bar{\gamma}$ to $\underline{\gamma}$. For paths γ_1, γ_2 such that $\bar{\gamma}_1 = \underline{\gamma}_2$, we write $\gamma_1 \gamma_2$ for the path obtained by their concatenation. Consider the set of loops

$$L(\Sigma) = \{\gamma \in P(\Sigma) : \underline{\gamma} = \bar{\gamma}\}.$$

Given a subset Γ of $P(\Sigma)$ which is closed under reversion and concatenation, we call a function $h : \Gamma \rightarrow G$ *multiplicative* if

$$h_{\gamma^{-1}} = h_{\gamma}^{-1}, \quad h_{\gamma_1 \gamma_2} = h_{\gamma_2} h_{\gamma_1}$$

for all γ and for all γ_1, γ_2 with $\bar{\gamma}_1 = \underline{\gamma}_2$. We denote the set of such multiplicative functions by $\mathcal{M}(\Gamma, G)$.

We say that $\mathbb{G} = (V, E)$ is an *embedded graph* in Σ if E is a finite subset of $P(\Sigma)$, V is the set of endpoints of paths in E , and each path of E have no intersection or self-intersection except at its endpoints. Then we write $P(\mathbb{G})$ for the subset of $P(\Sigma)$ obtained by concatenations of the edges E of \mathbb{G} and their reversions. Write F for the finite set of connected components of $\Sigma \setminus \{e^* : e \in E\}$, where e^* denotes the range of e . We say that an embedded graph \mathbb{G} is a *discretization* of Σ if each face $f \in F$ is a simply connected domain in Σ .

A random process $H = (H_{\gamma} : \gamma \in P(\Sigma))$ (on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$) taking values in G is a *Yang–Mills holonomy field* if

- (a) H is multiplicative, that is, $H(\omega) \in \mathcal{M}(P(\Sigma), G)$ for all $\omega \in \Omega$,
- (b) for any discretization $\mathbb{G} = (V, E, F)$ of Σ and all $h \in \mathcal{M}(P(\mathbb{G}), G)$,

$$\mathbb{P}(H_e \in dh_e \text{ for all } e \in E) = \frac{\prod_{f \in F} p_{|f|}(h_f)}{p_T(1)} \prod_{e \in E} dh_e, \quad (2)$$

- (c) for all paths $\gamma \in P(\Sigma)$ and any sequence of paths $(\gamma(n))_{n \in \mathbb{N}}$ in $P(\Sigma)$ converging to γ in length with fixed endpoints,¹

$$H_{\gamma(n)} \rightarrow H_\gamma \quad \text{in probability.}$$

Here, for each face f , we have written $|f|$ for the area of f and we have chosen a loop $\gamma \in L(\mathbb{G})$ parametrizing the boundary of f and set $h_f = h_\gamma$. The invariance properties of Haar measure and the heat kernel under inversion and conjugation guarantee that the expression (2) for the finite-dimensional distributions of H does not depend on the orientations of the edges, nor on the choice of loops bounding the faces.

Define coordinate functions $H_\gamma : \mathcal{M}(P(\Sigma), G) \rightarrow G$ by $H_\gamma(h) = h_\gamma$ and define a σ -algebra \mathcal{C} on $\mathcal{M}(P(\Sigma), G)$ by

$$\mathcal{C} = \sigma(H_\gamma : \gamma \in P(\Sigma)).$$

Then $(H_\gamma : \gamma \in P(\Sigma))$ is a multiplicative random process on $(\mathcal{M}(P(\Sigma), G), \mathcal{C})$. We use the same notation $(H_\gamma : \gamma \in P(\Sigma))$ both for this coordinate process, always identified as such, and also, more generally, for any multiplicative random process.

Our basic object of study is the *Yang–Mills measure* YM provided by the following theorem of Lévy [30, Theorem 2.62], building on earlier work of Driver [14] and Sengupta [39]. Note that, by uniqueness and the invariance properties of (2), the measure YM is invariant under area-preserving diffeomorphisms of Σ , so the relevant data from Σ are just its genus and the total area T .

Theorem 2.1. *There is a unique probability measure YM on $(\mathcal{M}(P(\Sigma), G), \mathcal{C})$ under which the coordinate process $(H_\gamma : \gamma \in P(\Sigma))$ is a Yang–Mills holonomy field.*

2.2 Embedded Brownian bridges

In each Yang–Mills holonomy field $H = (H_\gamma : \gamma \in P(\mathbb{S}_T))$ over the sphere \mathbb{S}_T , there are many embedded Brownian bridges in G from 1 to 1 in time T , as we now describe. Recall that a random process $B = (B_t : t \in [0, T])$ taking values in G is a Brownian bridge from 1 to 1 in time T if

- (a) B is continuous, that is, $B(\omega) \in C([0, T], G)$ for all $\omega \in \Omega$,
- (b) for all $n \in \mathbb{N}$, all $g_1, \dots, g_{n-1} \in G$ and all increasing sequences (t_1, \dots, t_{n-1}) in $(0, T)$, setting $g_0 = g_n = 1$ and $t_0 = 0$ and $t_n = T$ and writing $t_k = s_1 + \dots + s_k$,

$$\mathbb{P}(B_{t_k} \in dg_k \text{ for } k = 1, \dots, n-1) = \frac{\prod_{i=1}^n p_{s_i}(g_i g_{i-1}^{-1})}{p_T(1)} \prod_{k=1}^{n-1} dg_k.$$

Choose a point x in \mathbb{S}_T and let P be a tangent plane to \mathbb{S}_T at x , considered as embedded in \mathbb{R}^3 . Choose a line L in P through x and rotate \mathbb{S}_T once around L . The intersections of P with \mathbb{S}_T , which are a nested family of circles, may be considered as a family in $L(\mathbb{S}_T)$ all having x as their starting point and endpoint. We can parametrize this family of loops as $(l(t) : t \in [0, T])$ so that the domain inside $l(t)$ has area t for all T . Then, for all $n \in \mathbb{N}$ and all sequences (t_1, \dots, t_{n-1}) in $(0, T)$,

¹This means that the endpoints are fixed, the length of $\gamma(n)$ converges to the length of γ and $\sup_{t \in [0, 1]} d(\gamma_t(n), \gamma_t) \rightarrow 0$ for some parametrizations.

the loops $l(t_1), \dots, l(t_{n-1})$ are the edges of a discretization of \mathbb{S}_T . Define a random process $\beta = (\beta_t : t \in [0, T])$ in G by

$$\beta_t = H_{l(t)}.$$

It is straightforward to deduce from property (b) of H that the finite-dimensional distributions of β satisfy condition (b) for the Brownian bridge. Hence, by standard arguments, β has a continuous version, B say, which is a Brownian bridge in G from 1 to 1 in time T . The reader will see many ways to vary this construction while still obtaining a Brownian bridge.

2.3 The master field on the sphere

In this subsection we introduce the limit object for our main theorem, which is a certain function on loops

$$\Phi_T : L(\mathbb{S}_T) \rightarrow \mathbb{C}$$

known in the physics literature as the master field on the sphere.

We say that $l \in L(\mathbb{S}_T)$ is a *regular loop* if there exists a non-negative integer n and an embedded graph $\mathbb{G}^l = (V, E)$ having $n + 1$ vertices, one of degree 2 and the others all of degree 4, such that l starts from the vertex of degree 2 and is the concatenation of all $2n + 1$ edges, with transverse self-intersections at all vertices of degree 4. Note that a regular loop l determines its embedded graph \mathbb{G} uniquely.

Given a regular loop l , and a point v of self-intersection of l , there are two regular loops $l_{v,1}$ and $l_{v,2}$ starting from v , obtained by following the loop l until it first returns to v , both having fewer self-intersections than l . We say that a smooth map

$$\theta : (-\eta, \eta) \times \mathbb{S}_T \rightarrow \mathbb{S}_T$$

for some $\eta > 0$, is a Makeenko-Migdal flow at (v, l) if

1. $\theta(0, x) = x$ for all x ,
2. $\theta(t, \cdot)$ is a diffeomorphism of Σ for all t ,
3. for any face F of the embedded graph \mathbb{G}^l ,

$$\frac{d}{dt} |\theta(t, F)| = \begin{cases} 0, & \text{if } v \text{ does not belong to the boundary of } F \\ 1, & \text{if the edges bounding } F \text{ at } v \text{ have the same orientation} \\ -1, & \text{if they have different orientations.} \end{cases}$$

The following result characterizes, up to one function on $[0, T]$, the master field Φ_T on loops in the sphere of area T , which is the limit object for our main result. Let us say that two loops l, l' are equivalent and write $l \sim l'$, when l can be obtained from l' by adding or deleting loops of the form $\gamma\gamma^{-1}$.

Proposition 2.2. *For all $T \in (0, \infty)$, let $m : [0, T] \rightarrow \mathbb{C}$, be a function with continuous derivative, with $m(t) = m(T - t)$ for all $t \in [0, T]$ and $m(0) = 1 = m(T)$. There is at most one function $\Phi_m : L(\mathbb{S}_T) \rightarrow \mathbb{C}$ with the following properties:*

- (a) Φ_m is continuous on $L(\mathbb{S}_T)$,
- (b) for any regular loop l having a point of self-intersection point v and θ a Makeenko-Migdal flow at (l, v)

$$\left. \frac{d}{dt} \right|_{t=0} \Phi_m(\theta(t, l)) = \Phi_m(l_{v,1}) \Phi_m(l_{v,2}),$$

- (c) For any $l, l' \in L(\mathbb{S}_T)$, with $l \sim l'$, $\Phi_m(l) = \Phi_m(l')$.
(d) for any simple loop l whose complement in \mathbb{S}_T has a component of area a , then

$$\Phi_m(l) = m(a).$$

We shall prove this result in section 3.4. The formula (b) is called the *Makeenko-Migdal equation*. When $m : [0, T] \rightarrow \mathbb{C}$ is chosen appropriately, the field Φ_T can be approximated by traces of matrices and has a positivity property, that we shall state in Theorem 2.4 below.

Let μ_T be the unique probability measure on \mathbb{R} minimizing the functional

$$\mathcal{I}_T(\mu) = \int_{\mathbb{R}^2} \left(\frac{(x^2 + y^2)T}{2} - 2 \log |x - y| \right) \mu(dx) \mu(dy),$$

over the set $\{\mu \in \mathcal{M}_1(\mathbb{R}) : \mu \leq \text{Leb}\}$. By construction, μ_T has a density $\rho_T : \mathbb{R} \rightarrow [0, 1]$ with respect to Lebesgue measure. Its Stieltjes transform is defined for any $z \in \mathbb{C} \setminus \text{supp}(\mu_T)$ as

$$G_{\mu_T}(z) = \int_{\mathbb{R}} \frac{\mu_T(dx)}{z - x}.$$

Setting for any $a \in [0, T]$,

$$m_T(a) = \int_{\mathbb{R}} \cosh\left((a - \frac{T}{2})x\right) \sin(\pi \rho_T(x)) \frac{dx}{\pi}$$

we define a smooth function on $[0, T]$, satisfying the conditions of proposition 2.2. The prove the following in section 2.4.

Theorem 2.3. *There exists a function $\Phi_T = \Phi_{m_T}$, satisfying the conditions of proposition 2.2.*

We call Φ_T the *master field on \mathbb{S}_T* . It enjoys the following properties. Recall that for any point $p \in \mathbb{S}_T$, the subset $L_p(\mathbb{S}_T)$ of loops in $L(\mathbb{S}_T)$ with base point p , is endowed with a product rule, given by concatenation and an identity given by the constant loop at p , which makes it a monoïd. We denote by $\mathcal{A}_{p,T}$ the algebra of complex-valued functions on $L_p(\mathbb{S}_T)$ with finite support, endowed with the convolution product. We shall identify any loop $l \in L_p(\mathbb{S}_T)$ with the Dirac function $\delta_l \in \mathcal{A}_{p,T}$. This algebra is endowed with a skew Hermitian involution $*$: $\mathcal{A}_{p,T} \rightarrow \mathcal{A}_{p,T}$, that maps any l , with $l \in L_p(\mathbb{S}_T)$ viewed as an element of $\mathcal{A}_{p,T}$, to $l^{-1} \in \mathcal{A}_{p,T}$. We denote by 1 the Dirac function at the constant loop of $L_p(\mathbb{S}_T)$. We shall identify any loop $l \in L_p(\mathbb{S}_T)$ with the element of $\mathcal{A}_{p,T}$, given by the Dirac function at l . The prove the following in section 2.4.

Theorem 2.4. *The master field on \mathbb{S}_T satisfies the following properties. For any $p \in \mathbb{S}_T$,*

1. For any $l \in L_p(\mathbb{S}_T)$, $\Phi_T(ll^*) = 1$.
2. For any $x, y \in \mathcal{A}_{p,T}$, $\Phi_T(xy) = \Phi_T(yx)$.
- 2' For any path $\gamma \in P(\mathbb{S}_T)$ and $x \in \mathcal{A}_{\gamma,T}$, $\Phi_T(\gamma^{-1}x\gamma) = \Phi_T(x)$.
3. For any area-preserving diffeomorphism $\psi : \mathbb{S}_T \rightarrow \mathbb{S}_T$ and any $l \in L_p(\mathbb{S}_T)$,

$$\Phi_T(\psi(l)) = \Phi_T(l).$$

4. For any $x \in \mathcal{A}_{p,T}$, $\Phi_T(xx^*) \geq 0$.

Let us comment these properties. The point 2' is equivalent to the point 2 together with (c) of proposition 2.2. The points 1,2 and 4 imply the condition of the Gelfand-Naimark-Segal Theorem (see for instance [10, Chap XV, section 6]): for any $p \in \mathbb{S}_T$, there exists a Hilbert space $(\mathcal{H}_{p,\Phi_T}, \langle \cdot, \cdot \rangle_p)$, endowed with a left-action of $\mathcal{A}_{p,T}$ and a vector $\xi_p \in \mathcal{H}_{p,T}$, such that

$$\Phi_T(a) = \langle a\xi_p, \xi_p \rangle_p.$$

Note that the point 1 of the last Theorem ensures that for any $l \in L_p(\mathbb{S}_T)$, the action of l on $(\mathcal{H}_{p,T}, \langle \cdot, \cdot \rangle_p)$ is unitary; we denote by $\nu_l \in \mathcal{M}_1(\mathbb{U})$ its spectral measure, that is the unique measure satisfying for any $n \in \mathbb{Z}$,

$$\Phi_T(l^n) = \int_{\mathbb{U}} \omega^n \nu_l(\omega). \quad (3)$$

The points 3 and 4 yield the following. For any path $\gamma \in P(\mathbb{S}_T)$, there is a unitary operator $\iota_\gamma : \mathcal{H}_{\underline{\gamma}} \rightarrow \mathcal{H}_{\overline{\gamma}}$ satisfying $\iota(\xi_\gamma) = \xi_{\overline{\gamma}}$, whereas for any loop $l \in L(\mathbb{S}_T)$, $\iota_l = \text{Id}_{\mathcal{H}_{\ell,T}}$. The point 4 implies that for any area-preserving diffeomorphism θ of \mathbb{S}_T and any $p \in \mathbb{S}_T$, there is an isometry $\Psi_\theta : \mathcal{H}_p \rightarrow \mathcal{H}_{\theta(p)}$.

Remark. Let us note that the first two points could be proven for any field Φ_m , with m satisfying the conditions of proposition 2.2. It would be interesting to determine whether the conditions (a) and (b) of proposition 2.2 together with the one of Theorem 2.4 and that Φ_T is real valued, characterize uniquely Φ_T .

Remark. It is proved in [24, 29] that the relation \sim appearing in (c) of proposition 2.2 can be enhanced into a relation with larger orbits, such that the quotient of $L_p(\mathbb{S}_T)$ is a group for any $p \in \mathbb{S}_T$. Following [29], it could be shown that Theorems 2.2 and 2.4 hold true with this modified definition.

2.4 Convergence to the master field

For $g \in U(N)$, write $\text{tr}(g)$ for the normalized trace, given by

$$\text{tr}(g) = \frac{1}{N} \sum_{i=1}^N g_{ii}.$$

The following is our main result.

Theorem 2.5. *Let $T \in (0, \infty)$. Let $(H_\gamma : \gamma \in P(\mathbb{S}_T))$ be a Yang-Mills holonomy field in $U(N)$ and let $(\Phi_T(l) : l \in L(\mathbb{S}_T))$ be the master field on \mathbb{S}_T . Then, for all loops $l \in L(\mathbb{S}_T)$,*

$$\text{tr}(H_l) \rightarrow \Phi_T(l)$$

in probability as $N \rightarrow \infty$.

The organization of the proof is explained at the end of this section. Let us discuss some corollaries.

Proof of Theorem 2.3 and 2.4. Let $\Phi : L(\mathbb{S}_T) \rightarrow \mathbb{C}$ be the limit of Theorem 2.5. According to Proposition 3.9, Φ is continuous in length with fixed endpoints. Since almost surely $(H_l)_{l \in L(\mathbb{S}_T)}$ is multiplicative, if $l, l' \in L(\mathbb{S}_T)$ with $l \sim l'$, then $\Phi(l) =$

$\Phi(l')$. Now Theorem 3.1 and Proposition 4.2 imply that Φ satisfies all conditions of Proposition 2.2 with $m = m_T$, which proves Theorem 2.3. As for any area-preserving diffeomorphism ψ of \mathbb{S}_T , $\mathbb{E}[\text{tr}(H_l)] = \mathbb{E}[\text{tr}(H_{\psi(l)})]$, $\Phi_T(\psi(l)) = \Phi(l)$. What is more for any $p \in \mathbb{S}_T$ and $x = \sum_{i=1}^n \alpha_i l_i \in \mathcal{A}_p$, let us set $H_x = \sum_{i=1}^n \alpha_i H_{l_i}$. Then, for any $N \geq 1$, $\text{tr}(H_{xx^*}) = \text{tr}(H_x H_x^*) \geq 0$. Therefore, $\Phi_T(xx^*) \geq 0$. The other assertions are a consequence of the multiplicativity of $(H_\gamma)_{\gamma \in \mathcal{P}(\mathbb{S}_T)}$. \square

For any integer N and any loop $l \in \mathcal{L}(\mathbb{S}_T)$, setting

$$\nu_l^N = \frac{1}{N} \sum_{\omega \in \text{spec}(H_l)} \delta_\omega$$

defines a probability measure on the unit circle \mathbb{U} .

Corollary 2.6. *Under the assumption of the last theorem, for any $l \in \mathcal{L}(\mathbb{S}_T)$, ν_l^N converges weakly as $N \rightarrow \infty$ towards ν_l , defined by (3).*

Proof. Indeed, for all $n \in \mathbb{Z}$, by multiplicativity of $(H_l)_{l \in \mathcal{L}(\mathbb{S}_T)}$

$$\int_{\mathbb{U}} \omega^n \nu_l^N(\omega) = \text{tr}(H_l^n) = \text{tr}(H_{l^n}),$$

where $l^n \in \mathcal{L}(\mathbb{S}_T)$ is the loop obtained by $n - 1$ concatenations of l with itself. Therefore, by Theorem 2.5, the latter moment converges in probability towards $\Phi_T(l^n)$. \square

Free unitary Brownian loop: A corollary of Theorem 2.5 is that the Brownian bridge $(B_t)_{0 \leq t \leq T}$ in $U(N)$, from 1 to 1 in time T , converges in non-commutative distribution as $N \rightarrow \infty$. Consider the unital algebra

$$\mathcal{A}_T = \mathbb{C}\langle X_t, X_t^{-1}, 0 \leq t \leq T \rangle$$

of non-commutative polynomials

$$P = p(X_t, X_t^{-1} : 0 \leq t \leq T).$$

Write $*$ for the skew-hermitian and anti-multiplicative involution on \mathcal{A}_T such that $X_t^* = X_t^{-1}$ for all t . A non-commutative distribution on $(\mathcal{A}_T, *)$ is the data of a positive trace on this space, that is a linear functional such that for any $x, y \in \mathcal{A}_T$, $\tau(xy) = \tau(yx)$ and $\tau(xx^*) \geq 0$. For any $N \geq 1$, we define a random positive trace τ_N on \mathcal{A}_T setting

$$\tau_N(P) = \text{tr}(p(B_{t,T}, B_{t,T}^{-1} : 0 \leq t \leq T)).$$

Corollary 2.7. *For all $T \geq 0$, there is a deterministic trace τ on \mathcal{A}_T such that, for any $P \in \mathcal{A}_T$, $\tau_N(P) \rightarrow \tau(P)$ in probability as $N \rightarrow \infty$.*

Proof. It is sufficient to prove the statement for any monomial $P = X_{t_1}^{\varepsilon_1} \dots X_{t_n}^{\varepsilon_n}$, with $\varepsilon_1, \dots, \varepsilon_n \in \{1, -1\}$. Then, realizing $(B_{t,T})_{0 \leq t \leq T}$ as in section 2.2, $\text{tr}(P(B_{t,T}, B_{t,T}^*)) = \text{tr}(H_{l(t_1)}^{\varepsilon_1} \dots H_{l(t_n)}^{\varepsilon_n}) = \text{tr}(H_l)$, with $l = l(t_1)^{\varepsilon_1} \dots l(t_n)^{\varepsilon_n} \in \mathcal{L}(\mathbb{S}_T)$. Therefore, the convergence follows from Theorem 2.5 and the positivity from the same argument as in the proof of Theorem 2.4. \square

We call the non-commutative distribution $(\mathcal{A}_T, *, \tau)$ the *free unitary Brownian loop* and denote the canonical process by $(b_{t,T})_{0 \leq t \leq T}$. The spectrum of its time-marginals can be characterized as follows.

Proposition 2.8. *Let us assume that $(B_{t,T})_{0 \leq t \leq T}$ is a Brownian bridge on $U(N)$ and for any $0 \leq t \leq T$, denote by $\nu_{t,T}^N$ the empirical measure of eigenvalues of $B_{t,T}$. Then, $\nu_{t,T}^N$ converges in probability towards a measure $\nu_{t,T}$ such that*

$$\tau(b_{t,T}^n) = \nu_{t,T}(\omega^n) = \int_{\text{supp}(\mu_T) \cap \text{supp}(1-\mu_T)} \cosh(n(t - \frac{T}{2})) \sin(n\rho_T(x)) \frac{dx}{n\pi}.$$

Proof. Let us realize a Brownian bridge $(B_{t,T})_{0 \leq t \leq T}$ as in section 2.2. Then, for any $n \in \mathbb{Z}$, $\int_{\mathbb{U}} \omega^n \nu_{t,T}(d\omega) = \text{tr}(H_{l(t)}^n) = \text{tr}(H_{l(t)^n})$. The statement follows from Proposition 4.2 and Lemma 4.4. \square

We shall see in Section 5 that in the subcritical regime $T \leq \pi^2$, the free unitary loop has the same commutative distribution as the exponential of a free Brownian loop on $i\mathbb{R}$ with the same lifetime, that is, that the spectrum of $b_{t,T}$ for any $t \leq T \leq \pi^2$, is the push-forward of a Wigner law by the exponential mapping to the circle. Nonetheless, we shall see that their non-commutative distribution differ.

The proof of Theorem 2.5 will occupy much of the rest of the paper. The strategy is to prove first convergence for simple loops, then to show that this implies the general case. For the first part, we use harmonic analysis in $U(N)$ to express means and covariances of $\text{tr}(H_l)$ for simple loops l in terms of a discrete Coulomb gas, whose asymptotics as $N \rightarrow \infty$ are relatively accessible. This is done in Section 4. For the extension to all loops, we use the Makeenko–Migdal equations to build an understanding of $\text{tr}(H_l)$ for progressively more complex loops. This second stage is done in Section 3.

Proof of Theorem 2.5. The combination of the propositions 4.2, 3.7, 3.8 and 3.9 yields the result. \square

3 Makeenko–Migdal equations

3.1 Wilson loops

For any loop $l \in \mathcal{L}(\mathbb{S}_T)$, the random variable

$$W_l = \text{tr}(H_l)$$

on $(\mathcal{M}_N(P(\mathbb{S}_T)), \mathcal{C})$ is called a *Wilson loop*. If l is simple then W_l is also called simple. A *4-map* on \mathbb{S}_T is an embedded graph $\mathbb{G} = (\mathbb{V}, \mathbb{E}, \mathbb{F})$ on \mathbb{S}_T , without double edges, with vertices of degree 4 or 2, considered up to orientation preserving homeomorphism². A *combinatorial loop* is a loop in a 4-map \mathbb{G} without backtracking. We denote the set of such combinatorial loops of \mathbb{G} by $\mathfrak{L}(\mathbb{G})$. A *combinatorial skein* of \mathbb{G} is a collection of elements of $\mathfrak{L}(\mathbb{G})$, such that each edge of \mathbb{G} is used at most once. We denote their set by $\mathfrak{S}(\mathbb{G})$.

²Let us emphasize that embedded graphs are not necessarily connected, and possibly have faces which are not simply connected.

Lipschitz mapping of a graph: When $\mathbb{G} = (V, E)$ is a graph, a Lipschitz mapping \mathcal{G} of \mathbb{G} towards \mathbb{S}_T is a pair given by a collection of points $(p_v)_{v \in V}$ of \mathbb{S}_T and a collection of paths $(\gamma_e)_{e \in E^+}$ of $P(\mathbb{S}_T)$, such that $\gamma_{e^{-1}} = \gamma_e^{-1}$ and for any $(x, y), (y, z) \in E^+$, $\bar{\gamma}_{(x,y)} = p_y = \underline{\gamma}_{(y,z)}$. We denote by $\text{Lip}(\mathbb{G})$ the space of Lipschitz mapping of \mathbb{G} into \mathbb{S}_T and endow it with the product topology of length^3 on $P(\mathbb{S}_T)$. When $r \in V$ and $p \in \mathbb{S}_T$ are fixed, we denote by $\text{Lip}_{r,p}(\mathbb{G})$ the subset of $\text{Lip}(\mathbb{G})$ given by collections (p, γ) , such that $p_r = p$. The drawing into $\mathcal{G} = (p, \gamma) \in \text{Lip}(\mathbb{G})$, of a path going a sequence of vertices $x_1 x_2 \dots x_n$ of \mathbb{G} , is the concatenation $\gamma_{x_1 x_2} \gamma_{x_2 x_3} \dots \gamma_{x_{n-1} x_n} \in P(\mathbb{S}_T)$.

Remark. With this convention, an embedded graph as defined page 4 is a Lipschitz mapping of some graph.

Given a 4-map \mathbb{G} , we set

$$\Delta_{\mathbb{G}}(T) = \left\{ a \in \mathbb{R}_+^{\mathbb{F}} : \sum_{F \in \mathbb{F}} a_F = T \right\}.$$

For each $a \in \text{Int}(\Delta_{\mathbb{G}}(T))$, there exists an embedding⁴ of \mathbb{G} into \mathbb{S}_T in which each face $F \in \mathbb{F}$ is embedded with area a_F . We denote the equivalence class of such graphs up to area-preserving diffeomorphism by $\mathbb{G}(a)$. When $a \in \partial \Delta_{\mathbb{G}}(T)$, $\mathbb{G}(a)$ denotes the set of Lipschitz mapping $\mathcal{G}_a \in \text{Lip}(\mathbb{G})$ of \mathbb{G} such that there is $r \in V$, $p \in \mathbb{S}_T$, with $\mathcal{G}_a \in \text{Lip}_{r,p}(\mathbb{S}_T)$, a sequence $(a_n)_{n \geq 1}$ of $\text{Int}(\Delta_{\mathbb{G}}(T))$ with $a_n \rightarrow a$, and $\mathcal{G}_n \in \mathbb{G}(a_n) \cap \text{Lip}_{r,p}(\mathbb{S}_T)$, with $\mathcal{G}_n \rightarrow \mathcal{G}_a$. When \mathbb{G}, \mathbb{G}' are two combinatorial embedded graphs with $\mathbb{G} \subset \mathbb{G}'$, we define the affine map $\iota_{\mathbb{G}}^{\mathbb{G}'} : \Delta_{\mathbb{G}'}(T) \rightarrow \Delta_{\mathbb{G}}(T)$ by

$$(\iota_{\mathbb{G}}^{\mathbb{G}'} a)_F = \sum_{F' \in \mathbb{F}': F' \subset F} a_{F'}.$$

For any combinatorial embedded graph \mathbb{G} and $a \in \Delta_{\mathbb{G}}(T)$, the set of loops in $L(\mathbb{S}_T)$ obtained by drawing combinatorial loops of $\mathfrak{L}(\mathbb{G})$, in graphs belong to $\mathbb{G}(a)$, is denoted by $L(\mathbb{G}, a)$. For $\mathfrak{s} = \{l_1, \dots, l_k\} \in \mathfrak{S}(\mathbb{G})$, we define a function w_{l_1, \dots, l_k} on $\Delta_{\mathbb{G}}(T)$ by setting

$$w_{\mathfrak{s}}(a) = \mathbb{E}_{\text{YM}_{N,T}}(\text{tr}(H_{l_1}) \dots \text{tr}(H_{l_k}))$$

where $l_1, \dots, l_k \in L(\mathbb{G}, a)$ are drawing of respectively l_1, \dots, l_k . For $l \in \mathfrak{L}(\mathbb{G})$, denote by $\mathbb{G}_l = (\mathbb{V}_l, \mathbb{E}_l, \mathbb{F}_l)$ the 4-map explored by l and, for $T > 0$, set $\Delta_l(T) = \Delta_{\mathbb{G}_l}(T)$. Similarly, for $\{l, l'\} \in \mathfrak{S}(\mathbb{G})$, denote by $\mathbb{G}_{l,l'}$ the smallest embedded graph containing l and l' . Write \mathbb{V}_l^s for the points of transverse self intersection of l and write $\mathbb{V}_{l,l'}^m$ for the points of transverse mutual intersection of l and l' .

3.2 Makeenko–Migdal equations for a pair of loops

We shall give now a version of the Makeenko–Migdal equations – that already appeared in 2.2 for the master field – for the Yang–Mills measure with unitary structure groups. Let \mathbb{G} be a regular 4-map in a compact surface of volume T , and let $\{l, l'\} \in \mathfrak{S}(\mathbb{G})$. Given $v \in \mathbb{V}_l^s$, let us write $l_{v,1}$ and $l_{v,2}$ for the two loops based at v following the strand of l , starting with the two outgoing edges of l at v and stopping

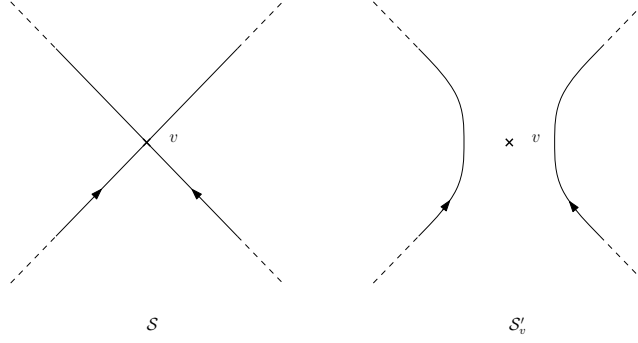


Figure 1: Desingularization operation

at their first return to v (see figure 1). On the other hand, given $v \in \mathbb{V}_{\mathfrak{l}, \mathfrak{l}'}$, let us write $\mathfrak{l} \circ_v \mathfrak{l}'$ for the loop based at v obtained by concatenation of the loops \mathfrak{l} and \mathfrak{l}' rerooted at v . In each of these cases, we define a (constant) vector field μ_v on the interior of $\Delta_{\mathbb{G}}(T)$, setting

$$\mu_v = \partial_{a_{F_1}} - \partial_{a_{F_2}} + \partial_{a_{F_3}} - \partial_{a_{F_4}} \quad (4)$$

where F_1, \dots, F_4 are the faces of \mathbb{G} around v , labelled cyclicly and such that F_1 bounds two outgoing or two incoming edges at v . The Makeenko–Migdal equations for a pair of loops⁵ were proved for any compact surface by Driver et al. [12, 13].

Theorem 3.1. *For any regular 4-map \mathbb{G} and any $\{\mathfrak{l}, \mathfrak{l}'\} \in \mathfrak{S}(\mathbb{G})$, we have*

$$\mu_v w_{\mathfrak{l}} = w_{\mathfrak{l}_{v,1}, \mathfrak{l}_{v,2}}, \quad \text{if } v \in \mathbb{V}_{\mathfrak{l}}^s$$

and

$$\mu_v w_{\mathfrak{l}, \mathfrak{l}'} = \begin{cases} w_{\mathfrak{l}_{v,1}, \mathfrak{l}_{v,2}, \mathfrak{l}'}, & \text{if } v \in \mathbb{V}_{\mathfrak{l}}^s, \\ N^{-2} w_{\mathfrak{l} \circ_v \mathfrak{l}'}, & \text{if } v \in \mathbb{V}_{\mathfrak{l}, \mathfrak{l}'}^m. \end{cases}$$

3.3 Makeenko–Migdal vector field

For an embedded graph \mathbb{G} , without double edges, in an orientable compact closed surface Σ_g of genus g , consider the space

$$X_{\mathbb{G}} \simeq \mathbb{R}^{\mathbb{F}} / \mathbb{R} \cdot 1_{\mathbb{F}} \simeq \{1_{\mathbb{F}}\}^{\perp} \cap \mathbb{R}^{\mathbb{F}}$$

of translation invariant vector fields on $\Delta_{\mathbb{G}}(T)$, endowed with the restriction of the usual scalar product of $\mathbb{R}^{\mathbb{F}}$. For $\{\mathfrak{l}, \mathfrak{l}'\} \in \mathfrak{S}(\mathbb{G})$, consider the following subspaces of $X_{\mathbb{G}}$

$$\begin{aligned} \mathfrak{m}_{\mathfrak{l}}^{\mathbb{G}} &= \text{span}\{\mu_v : v \in \mathbb{V}_{\mathfrak{l}}^s\} \\ \mathfrak{m}_{\mathfrak{l}, \mathfrak{l}'}^{\mathbb{G}} &= \text{span}\{\mu_v : v \in \mathbb{V}_{\mathfrak{l}}^s \cup \mathbb{V}_{\mathfrak{l}'}^s \cup \mathbb{V}_{\mathfrak{l}, \mathfrak{l}'}^m\}. \end{aligned}$$

³for the latter topology, a sequence $(\gamma(n))$ converges if and only the length of $\gamma(n)$ converges to the length of γ and $\sup_{t \in [0,1]} d(\gamma_t(n), \gamma_t) \rightarrow 0$ for some parametrizations.

⁴as defined page 4.

⁵and also for skeins

We call an element of these spaces a Makeenko–Migdal vector. When \mathbb{G} is respectively equal to \mathbb{G}_l and $\mathbb{G}_{l,l'}$, we shall drop the superscript. In the case of the whole plane, the spaces \mathfrak{m}_l and $\mathfrak{m}_{l,l'}$ were described by Lévy [29, Lemma 6.28] (see also Dahlqvist [8, Lemma 21]). We now adapt Lévy’s argument to the case of the sphere. Let us denote by $\hat{\mathbb{G}} = (\hat{\mathbb{V}}, \hat{\mathbb{E}}, \hat{\mathbb{F}})$ the graph dual to \mathbb{G} , with vertices, edges and faces indexed by \mathbb{F} , \mathbb{E} and \mathbb{V} . For $v \in \mathbb{V}$, write Out_v for the set of outgoing edges at v , with the convention that if e is a looping edge at a vertex v , then $\text{Out}_v \supset \{e, e^{-1}\}$. For $e \in \mathbb{E}$, write $F_L(e)$ and $F_R(e)$ for the faces to the left and to the right of e .

The space $\Omega^1(\mathbb{G}, \mathbb{R})$ of discrete one forms on \mathbb{G} is the vector space of functions $\omega : \mathbb{E} \rightarrow \mathbb{R}$, satisfying $\omega(e^{-1}) = -\omega(e)$. The discrete differential operators d are the maps defined by

$$\begin{aligned} d : \mathbb{R}^{\mathbb{F}} &\longrightarrow \Omega^1(\mathbb{G}, \mathbb{R}) \\ \alpha &\longmapsto (e \mapsto \alpha(F_L(e)) - \alpha(F_R(e))) \end{aligned}$$

and

$$\begin{aligned} d : \Omega^1(\mathbb{G}, \mathbb{R}) &\longrightarrow \mathbb{R}^{\mathbb{V}} \\ \omega &\longmapsto (v \mapsto \sum_{e \in \text{Out}_v} \omega(e)). \end{aligned}$$

We denote by $B^1(\mathbb{G}, \mathbb{R}) = d(\mathbb{R}^{\mathbb{F}})$ and $Z^1(\mathbb{G}, \mathbb{R}) = \ker(d : \Omega^1(\mathbb{G}, \mathbb{R}) \rightarrow \mathbb{R}^{\mathbb{V}})$ the spaces of exact and closed discrete one-forms. Then $d^2 = 0$, so $B^1(\mathbb{G}, \mathbb{R}) \subset Z^1(\mathbb{G}, \mathbb{R})$. Set $H^1(\mathbb{G}, \mathbb{R}) = Z^1(\mathbb{G}, \mathbb{R})/B^1(\mathbb{G}, \mathbb{R})$. Recall that, if all faces of $\hat{\mathbb{G}}$ are simply connected, then $H^1(\mathbb{G}, \mathbb{R})$ has dimension $2g$. Given a path $\gamma = (e_1, \dots, e_k)$ in $\hat{\mathbb{G}}$, set

$$\omega(\gamma) = \sum_{k=1}^n \omega(e_k).$$

Then

$$B^1(\mathbb{G}, \mathbb{R}) = \{\omega \in \Omega^1(\mathbb{G}, \mathbb{R}) : \omega(l) = 0 \text{ for all } l \in \mathfrak{L}(\hat{\mathbb{G}})\}.$$

For any loop l in \mathbb{G} , we define a closed one-form $\delta_l \in Z^1(\mathbb{G}, \mathbb{R})$ setting for any edge $e \in \mathbb{E}$, $\delta_l(e) = 1$, if e is traversed forward by l , -1 , if it is traversed backwards and 0 , otherwise. When \mathbb{G} is embedded in the sphere, $B^1(\mathbb{G}, \mathbb{R}) = Z^1(\mathbb{G}, \mathbb{R})$ and for any surface, $\ker(d : \mathbb{R}^{\mathbb{F}} \rightarrow \Omega^1(\mathbb{G}, \mathbb{R})) = \mathbb{R}1_{\mathbb{F}}$. If $\Sigma = \mathbb{S}_2$, we can therefore define $n_l \in \mathbb{R}^{\mathbb{F}}/\mathbb{R}1_{\mathbb{F}}$ as the preimage of δ_l . For any face $F_{\infty} \in \mathbb{F}$, let $n_{F_{\infty}, l} \in \mathbb{Z}^{\mathbb{F}}$ be the function such that $dn_{F_{\infty}, l} = \delta_l$ and $n_{F_{\infty}, l}(F_{\infty}) = 0$. The value of $n_{F_{\infty}, l}$ at a face $F \in \mathbb{F}$ is given by the winding number around the face F of the loop l , where \mathbb{S} is viewed as the compactified plane, with the infinite point lying in F_{∞} .

Lemma 3.2. *Let \mathfrak{s} be a combinatorial skein with one or two loops lying in a graph \mathbb{G} embedded in the sphere \mathbb{S} . The Makeenko–Migdal vector space is given by*

$$\mathfrak{m}_{\mathfrak{s}}^{\mathbb{G}} = \{n_l, l \in \mathfrak{s}\}^{\perp} \cap X_{\mathbb{G}}.$$

Proof. We show equivalently that $\mathfrak{m}_{\mathfrak{s}}^{\perp} \cap X_{\mathbb{G}} = \{n_l, l \in \mathfrak{s}\}$. For any vertex $v \in \mathbb{V}_{\mathfrak{s}}$, let e_{nw}, e_{ne} and e_{sw}, e_{se} be respectively the left and right outgoing and ingoing edges at v , of a strand of \mathfrak{s} . Then, for any function $\alpha \in \mathbb{R}^{\mathbb{F}}$, $\langle \alpha, \mu_v \rangle = d\alpha(e_{se}) - d\alpha(e_{nw}) = d\alpha(e_{sw}) - d\alpha(e_{ne})$. On the one hand, for any $l \in \mathfrak{s}$, the latter right-hand-side vanishes when $d\alpha = \delta_l$, so that $n_l \in \mathfrak{m}_{\mathfrak{s}}^{\perp}$. On the other hand, if $\alpha \in \{\mu_v, v \in \mathbb{V}\}^{\perp} \cap X_{\mathbb{G}}$, then $d\alpha$ is constant along any loop of \mathfrak{s} so that there exist coefficients $(x_l)_{l \in \mathfrak{s}}$, with $d\alpha = \sum_{l \in \mathfrak{s}} x_l \delta_l$. Therefore, $\alpha = \sum_{l \in \mathfrak{s}} x_l n_l \in X_{\mathbb{G}}$. \square

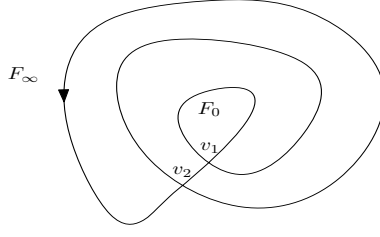


Figure 2: A maximally winding loop that is a drawing of wl_3 .

The following lemma is elementary but crucial in the argument. For any combinatorial loop \mathfrak{l} embedded in the sphere, let $F_\infty, F_0 \in \mathbb{F}_\mathfrak{l}$ be such that

$$n_\mathfrak{l}(F_0) - n_\mathfrak{l}(F_\infty) = \max\{n_\mathfrak{l}(F_1) - n_\mathfrak{l}(F_2) : F_1, F_2 \in \mathbb{F}_\mathfrak{l}\}. \quad (5)$$

For any $F, F' \in \mathbb{F}$, let us set

$$v_F(F') = \begin{cases} -1 & , \text{if } F' = F \\ \frac{n_\mathfrak{l}(F_0) - n_\mathfrak{l}(F)}{n_\mathfrak{l}(F_0) - n_\mathfrak{l}(F_\infty)} & , \text{if } F' = F_\infty \\ \frac{n_\mathfrak{l}(F) - n_\mathfrak{l}(F_\infty)}{n_\mathfrak{l}(F_0) - n_\mathfrak{l}(F_\infty)} & , \text{if } F' = F_0 \\ 0 & , \text{otherwise} \end{cases}$$

and for any $a \in \Delta_\mathfrak{l}(T)$ and $t \in [0, 1]$,

$$a_t = a + t \sum_{F \in \mathbb{F} \setminus \{F_\infty, F_0\}} a_F v_F. \quad (6)$$

Lemma 3.3. *i) For any combinatorial loop \mathfrak{l} , $F \in \mathbb{F}_\mathfrak{l}$, $v_F \in \mathfrak{m}_\mathfrak{l}$.*

ii) For any $t \in [0, 1]$, $a_t \in \Delta_\mathfrak{l}(T)$, and $a_1(F) = 0$, if $F \notin \{F_0, F_\infty\}$.

Proof. By construction, for any $F \in \mathbb{F}_\mathfrak{l} \setminus \{F_0, F_\infty\}$, $v_F \in X_{\mathbb{G}_\mathfrak{l}} \cap \{n_\mathfrak{l}\}^\perp$ and v_F is non-negative but at F , so that the result follows. \square

Now, any loop $l \in L(\mathbb{G}, a)$, where $a \in \Delta_\mathbb{G}(T)$ is supported on two faces $F_0, F_\infty \in \mathbb{F}$ with $a_{F_0}, a_{F_\infty} > 0$, is valued in the boundary of a simply connected domain of finite length boundary; therefore, it can be reduced, by erasing backtracking edges, to a loop γ^n of $\mathbb{G}'(a(F_0), a(F_\infty))$, where \mathbb{G}' is an embedded graph with two faces and one edge γ . To conclude this section, we shall use another 'move' to transform such a loop into a simple loop. For any $n \in \mathbb{N}^*$, let wl_n be the combinatorial loop winding n times around a fixed point (see figure 2), such that the dual graph of \mathbb{G}_{wl_n} is a segment, and let $F'_0, F'_\infty \in \mathbb{G}_{wl_n}$ be as in (5). Then, (F'_0, F'_∞) is the only pair maximizing the distance in $\hat{\mathbb{G}}_{wl_n}$. A loop $l \in L(\mathbb{S}_T)$, with $\mathbb{G}^l \in \mathbb{G}_{wl_n}(b)$ for some $n \in \mathbb{N}$ and $b \in \Delta_{\mathbb{G}_{wl_n}}(T)$ is called *maximally winding*. With this notation, any $l \in L(\mathbb{G}, a)$, parametrized by a smooth curve, with a as above, belongs also to $L(\mathbb{G}_{wl_n}, a')$ where $n = n_{F_\infty, \mathfrak{l}}(F_0)$, $a'(F'_0) = a(F_0)$, $a'(F'_\infty) = a(F_\infty)$ and $a'(F) = 0$, if $F \in \mathbb{F}_{wl_n} \setminus \{F_0, F_\infty\}$.

Lemma 3.4. *Let $n \in \mathbb{Z}$ with $n \geq 1$ and let $a \in \Delta_{wl_{n+1}}(T)$. Then there exists $v \in \mathfrak{m}_{wl_{n+1}}$ such that $a + tv \in \Delta_{wl_{n+1}}(T)$ for all $t \in [0, 1]$ and $L(\mathbb{G}_{wl_{n+1}}, a + v) \subset L(\mathbb{G}_{wl_n}, a')$, for some $a' \in \Delta_{wl_n}(T)$.*

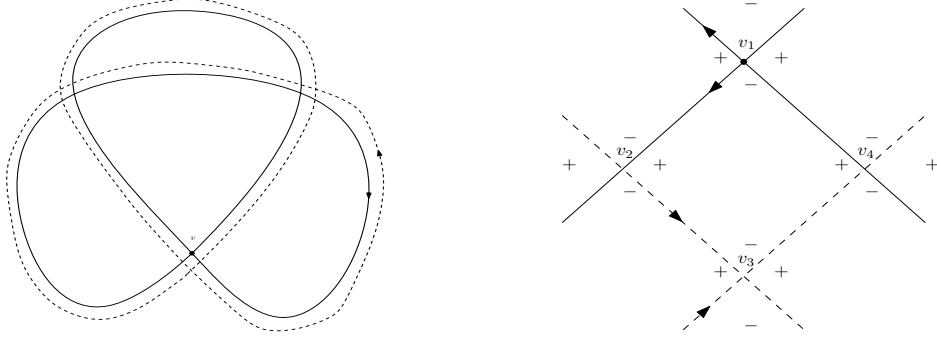


Figure 3: On the left-hand-side, the doubling of the graph \mathbb{G}_l of a combinatorial loop l drawn in plane lines; \tilde{l}^{-1} is drawn in dashed line. The right-hand-side pictures the decomposition of $\mathcal{D}(\mu_v)$ used in lemma 3.5, for the vertex v of \mathbb{G}_l represented by a dot.

Proof. Let $v_1 \in \mathbb{V}_{w_{l_{n+1}}}$ be the vertex belonging to the boundary of F_0 and v_2, \dots, v_n be the other vertices of $\mathbb{G}_{w_{l_{n+1}}}$, ordered by their time of first visit by w_{l_n} starting from v_1 . Then $\tilde{v} = \mu_{v_1} + \dots + \mu_{v_n} \in \mathfrak{m}_{w_{l_{n+1}}}$ is such that

$$\tilde{v}(F) = \begin{cases} -1 & , \text{if } F \in \{F_0, F_\infty\}, \\ 1 & , \text{if } \bar{F} \cap (\bar{F}_0 \cup \bar{F}_\infty) \neq \emptyset, \\ 0 & , \text{otherwise.} \end{cases}$$

Then $v = \min(a(F_0), a(F_\infty))\tilde{v}$ has the claimed properties. \square

Given a combinatorial loop l , consider the *double embedding* $\tilde{\mathbb{G}}_l$ obtained as follows. Embed l in \mathbb{S}_T and replace each embedded vertex v with four vertices v_1, v_2, v_3, v_4 , lying in the four corners around v , in a cyclic order, and add four edges forming a square around v . Replace each embedded edge (v, v') by two edges joining the two pairs of new vertices lying in the two faces neighboring (v, v') (see figure 3).

Then $\tilde{\mathbb{G}}_l$ is a 4-regular graph containing two embeddings of l , which we will denote by l and \tilde{l} , with \tilde{l} to the right of l . By Euler's relation,

$$\#\tilde{\mathbb{F}}_l = 2 + 4\#\mathbb{V}_l = 4\#\mathbb{F}_l - 6.$$

There is an injection $\iota : \mathbb{F}_l \rightarrow \tilde{\mathbb{F}}_l$ that identifies faces of \mathbb{G}_l with the faces of $\tilde{\mathbb{G}}_l$ that are not fully contracted, when retracting \tilde{l} to l . This latter induces an injective map

$$\mathcal{D} : \mathbb{R}^{\mathbb{F}_l} \rightarrow \mathbb{R}^{\tilde{\mathbb{F}}_l},$$

such that for any $h \in \mathbb{R}^{\mathbb{F}_l}$, and $F' \in \tilde{\mathbb{F}}_l$, $\mathcal{D}(h)(F') = h(F)$, if $F' = \iota(F)$ for some $F \in \mathbb{F}_l$, and 0, otherwise. It satisfies $\mathcal{D}(X_{\mathbb{G}_l}) \subset X_{\tilde{\mathbb{G}}_l}$ and for any $T > 0$, $\mathcal{D}(\Delta_{\mathbb{G}}(T)) \subset \Delta_{\tilde{\mathbb{G}}}(T)$.

Lemma 3.5. *The map \mathcal{D} satisfies*

$$\mathcal{D} : \mathfrak{m}_l \rightarrow \mathfrak{m}_{l, \tilde{l}} \cap \mathfrak{m}_{l, \tilde{l}^{-1}}.$$

Proof. For any $v \in \mathbb{V}_l$, let $(v_i)_{1 \leq i \leq 4}$ be the associated vertices of $\tilde{\mathbb{G}}_l$, in cyclic order with v_1 being an self-intersection point of l . Then (see figure 3),

$$\mathcal{D}(\mu_v^l) = \mu_{v_1}^{l, \tilde{l}} + \mu_{v_2}^{l, \tilde{l}} + \mu_{v_3}^{l, \tilde{l}} + \mu_{v_4}^{l, \tilde{l}} = \mu_{v_1}^{l, \tilde{l}^{-1}} - \mu_{v_2}^{l, \tilde{l}^{-1}} + \mu_{v_3}^{l, \tilde{l}^{-1}} - \mu_{v_4}^{l, \tilde{l}^{-1}}.$$

□

3.4 Reduction to the case of simple loops

In Proposition 4.2, we will get the following result for simple loops.

Proposition 3.6. *Let $(H_\gamma : \gamma \in P(\mathbb{S}_T))$ be a Yang-Mills holonomy field in $U(N)$. Denote by $(\Phi_T(l) : l \in L(\mathbb{S}_T))$ the master field on the sphere. Then $\text{tr}(H_l) \rightarrow \Phi_T(l)$ in probability as $N \rightarrow \infty$ for all simple loops $l \in \mathbb{S}_T$.*

In the following three propositions we extend this convergence to all loops, simple or not, thereby proving Theorem 2.5. We shall prove simultaneously Proposition 2.2.

Remark. We prove in Section 4 a stronger result, such that the following proposition 3.7 could be bypassed to prove Theorem 2.5. We include it here nonetheless, as it allows to prove Proposition 2.2.

Proposition 3.7. *i) Suppose that $\text{tr}(H_l) \rightarrow \Phi_T(l)$ in probability as $N \rightarrow \infty$ for all simple loops l . Then the same holds for all maximally winding loops.⁶*

ii) If Ψ and Φ satisfy the conditions of proposition 2.2, then $\Phi(l) = \Psi(l)$ for all maximally winding loop of \mathbb{S}_T .

Proof. Let us set $\Gamma = \Phi - \Psi$. Suppose inductively that the conclusions of i) and ii) hold for maximally winding loops having at most $n - 1$ self-intersections. The case $n = 1$ holds by assumption. Consider a maximally winding loop with n intersections, so that $l \in L(\mathbb{G}_{\text{wl}_{n+1}}, a)$ is a drawing of wl_{n+1} for some $a \in \Delta_{\text{wl}_{n+1}}(T)$. Let us choose $v \in \mathfrak{m}_{\text{wl}_{n+1}}$ and $a' \in \Delta_{\text{wl}_n}$ as in Lemma 3.4. Then, according to Theorem 3.1, for any $t \in (0, 1)$,

$$\frac{d}{dt} w_{\text{wl}_{n+1}}(a + tv) = \sum_{i=1}^n w_{(\text{wl}_{n+1})_{v_i, 1}, (\text{wl}_{n+1})_{v_i, 2}}(a + tv)$$

whereas the choice of v yields

$$w_{\text{wl}_{n+1}}(a_1) = w_{\text{wl}_n}(a').$$

Similarly, let us choose a family of diffeomorphisms $\theta : [0, 1] \times \mathbb{S}_T \rightarrow \mathbb{S}_T$ such that $|\theta(t, F)| = a_F + tv_F$, for all faces F of $\mathbb{G}^{\theta(t, l)}$, $\theta(0, l) = \text{Id}_{\mathbb{S}_T}$ and $\theta(t, l)$ converges in length with fixed endpoints, towards a maximally winding loop l_1 , with $n - 1$ intersections. Setting $l_t = \theta(t, l)$ and for $i \in [1, n]$, $v_i^t = \theta(t, v_i)$, by the condition (b) of Theorem 2.2, for all $t \in (0, 1)$,

$$\frac{d}{dt} \Gamma(l_t) = \sum_{i=1}^n (\Gamma((l_t)_{v_i^t, 1}) \Phi((l_t)_{v_i^t, 2}) - \Psi((l_t)_{v_i^t, 1}) \Gamma((l_t)_{v_i^t, 2})).$$

Since $l_1, (l_t)_{v_i^t, 1}, (l_t)_{v_i^t, 2}$ are regular maximally winding loops with less than $n - 1$ intersections, $\Gamma(l) = 0$. Moreover, for any integer $i \in [1, n]$, $(\text{wl}_{n+1})_{v_i, 1}$ and

⁶their definition is given above Lemma 3.4.

$(wl_{n+1})_{v_i,2}$ are equivalent to the combinatorial loops wl_i, wl_{n+1-i} ; let us denote this combinatorial skein by $\{l_i, r_i\}$. From inductive hypothesis, bounding $|\text{tr}(H_{(l_t)_{v_i^t,k}})|$, by 1, for $k \in \{1, 2\}$, $i \in [1, n]$, it follows that $wl_{i,r_i}(a + tv)$ converges towards $\Phi_T((l_t)_{v_i^t,1})\Phi_T((l_t)_{v_i^t,2})$. We deduce now by dominated convergence, bounding wl_{i,r_i} by 1 for any $i \in [1, n]$, that $w_{wl_{n+1}}$ converges as $N \rightarrow \infty$ towards

$$\Phi_T(l_1) - \sum_{i=1}^n \int_0^1 \Phi_T((l_t)_{v_i^t,1})\Phi_T((l_t)_{v_i^t,2})dt = \Phi_T(l),$$

where we applied the Makeenko-Migdal equation for the master field Φ_T . To conclude it is sufficient to show that $\text{cov}_N(\text{tr}(H_l), \overline{\text{tr}(H_l)}) \rightarrow 0$, as $N \rightarrow \infty$. Therefor, the argument used in the second part of the proof of Proposition 3.8 applies verbatim here, with l replaced by wl_{n+1} and the Makeenko-Migdal vector of Lemma 3.3 replaced by the one of Lemma 3.4. It is left as an exercise to the reader. \square

Proposition 3.8. *If one of following assertion holds true for all maximally winding loops l , then it does for all regular loops:*

- i) $\text{tr}(H_l) \rightarrow \Phi_T(l)$ in probability as $N \rightarrow \infty$.
- ii) $\Psi(l) = \Phi(l)$, where Ψ and Φ satisfy the conditions of proposition 2.2.

Proof. This proof has the same pattern as the previous one, using Lemma 3.3 instead of Lemma 3.4. Let us set $\Gamma = \Phi - \Psi$. In the case where l is simple or has a single point of intersection, l is maximally winding, so the desired convergence holds. Suppose inductively for $n \geq 1$ that i) and ii) holds true for all regular loops l having less than n points of intersection. Consider a regular loop l with $n + 1$ intersection points, with combinatorial loop l and area vector $a \in \Delta_l(T)$. Choose $(a_t)_{t \in [0,1]}$ as in Lemma 3.3. If $\sum_{F \in \mathbb{F} \setminus \{F_0, F_\infty\}} a_F v_F = \sum_{v \in \mathbb{V}_l} x_v \mu_v$, with $x \in \mathbb{R}^{\mathbb{V}_l}$, then, according to Makeenko-Migdal equations of Theorem 3.1, for any $t \in (0, 1)$,

$$\frac{d}{dt} w_l(a_t) = \sum_{v \in \mathbb{V}_l} x_v w_{l_{v,1}, l_{v,2}}(a_t)$$

whereas the choice of $(a_t)_{t \in [0,1]}$ yields

$$w_l(a_1) = w_{wl_n}(a'),$$

where $n = n_{F_\infty, l}(F_0)$ and for any $F' \in \mathbb{F}_{wl_n}$, $a'(F') = a_1(F)$, if $F \in \{F_0, F_\infty\}$ and 0 otherwise. Let us choose a family of diffeomorphisms $\theta : [0, 1] \times \mathbb{S}_T \rightarrow \mathbb{S}_T$ such that $|\theta(t, F)| = a_t$ for all faces F of $\mathbb{G}^{\theta(t, l)}$, $\theta(0, l) = \text{Id}_{\mathbb{S}_T}$ and $\theta(t, l)$ converges in length with fixed endpoints, towards a loop $l_1 \in L(\mathbb{S}_T)$ with a smooth parametrization, that is equivalent to a maximally winding loop, by erasing backtracking edges. Setting $l_t = \theta(t, l)$ and for $v \in \mathbb{V}_l$, $v^t = \theta(t, v)$, by the condition (b) of Theorem 2.2, for all $t \in (0, 1)$,

$$\frac{d}{dt} \Gamma(l_t) = \sum_{v \in \mathbb{V}_l} x_v (\Gamma((l_t)_{v^t,1})\Phi((l_t)_{v^t,2}) - \Psi((l_t)_{v^t,1})\Gamma((l_t)_{v^t,2})).$$

Since l_1 can be reduced to a maximally winding loop \bar{l}_1 and $(l_t)_{v_i^t,1}, (l_t)_{v_i^t,2}$ are regular loops with less than n intersections, $\Gamma(l) = 0$. Since, for any $v \in \mathbb{V}_l$, $l_{v,1}$ and $l_{v,2}$ have less than n intersection points, by inductive hypothesis, for any $a \in \Delta_l(T)$,

$w_{\mathfrak{l}_{v,1}, \mathfrak{l}_{v,2}}(a) \rightarrow w_{\mathfrak{l}_{v,1}}(a)w_{\mathfrak{l}_{v,2}}(a)$, as $N \rightarrow \infty$, whereas $w_{\mathfrak{l}_{v,1}, \mathfrak{l}_{v,2}}$ is uniformly bounded by 1. Besides, \tilde{l}_1 belongs also to $L(\mathbb{G}_{w_{l_1}}, a')$, where $n = n_{F_\infty, \mathfrak{l}}(F'_0)$ and for any $F \in \mathbb{F}_{w_n}$, $a'(F) = a_1(F)$, if $F \in \{F'_0, F'_\infty\}$ and 0 otherwise. Hence, by dominated convergence, as $N \rightarrow \infty$,

$$\begin{aligned} w_{\mathfrak{l}}(a) &= w_{w_n}(a') - \sum_{v \in \mathbb{V}_l} x_v \int_0^1 w_{\mathfrak{l}_{v,1}, \mathfrak{l}_{v,2}}(a_t) dt \\ &\rightarrow \Phi_T(l_1) - \sum_{v \in \mathbb{V}_l} x_v \int_0^1 \Phi_T((l_t)_{v^t, 1}) \Phi((l_t)_{v^t, 2}) dt = \Phi_T(l). \end{aligned}$$

Estimate on the variance: we shall prove now that the covariance of the Wilson loops $\text{cov}_N(\text{tr}(H_l), \overline{\text{tr}(H_l)})$ vanish as $N \rightarrow \infty$, which will conclude the proof. Note that, since H_l is unitary,

$$\text{cov}_N(\text{tr}(H_l), \overline{\text{tr}(H_l)}) = w_{\mathfrak{l}, \tilde{l}^{-1}}(\tilde{a}) - w_{\mathfrak{l}}(\tilde{a})w_{\tilde{l}^{-1}}(\tilde{a}),$$

where $(\mathfrak{l}, \tilde{l})$ are copies of \mathfrak{l} in the double graph⁷ $\tilde{\mathbb{G}}_l$ and $\tilde{a} = \mathcal{D}(a) \in \Delta_T(\tilde{\mathbb{G}}_l)$. For any $t \in [0, 1]$, set $\tilde{a}_t = \mathcal{D}(a_t) \in \Delta_{\tilde{\mathbb{G}}_l}(T)$. Recall that \tilde{l}_1 is a maximally winding loop obtained by erasing backtracking edges in l_1 . Therefore, by assumption,

$$\begin{aligned} |w_{\mathfrak{l}, \tilde{l}^{-1}}(\tilde{a}_1) - w_{\mathfrak{l}}(\tilde{a}_1)w_{\tilde{l}^{-1}}(\tilde{a}_1)| &= |\text{cov}_N(\text{tr}(H_{l_1}), \overline{\text{tr}(H_{l_1})})| = |\text{cov}_N(\text{tr}(H_{\tilde{l}_1}), \overline{\text{tr}(H_{\tilde{l}_1})})| \\ &= |w_{w_{l_1}, \tilde{w}_{l_1}^{-1}}(\tilde{a}') - w_{w_{l_1}} w_{\tilde{w}_{l_1}^{-1}}(\tilde{a}')| \rightarrow_{N \rightarrow \infty} 0, \end{aligned} \quad (7)$$

with $n \in \mathbb{N}^*$ and a' as in the first part of the proof and $\tilde{a}' = \mathcal{D}(a')$. Moreover, for any $t \in (0, 1)$, using Theorem 3.1 in $\tilde{\mathbb{G}}_l$, and the labeling of the vertices of $\tilde{\mathbb{G}}$ given in Lemma 3.5 (see figure 3),

$$\frac{d}{dt} w_{\mathfrak{l}, \tilde{l}^{-1}}(\tilde{a}_t) = \sum_{v \in \mathbb{V}_l} x_v \left(w_{\mathfrak{l}_{v,1}, \mathfrak{l}_{v,2}, \tilde{l}^{-1}} + w_{\mathfrak{l}, \tilde{l}_{v,3,1}^{-1}, \tilde{l}_{v,3,2}^{-1}} - N^{-2}(w_{\mathfrak{l}_{\mathfrak{o}_{v_2}} \tilde{l}^{-1}} + w_{\mathfrak{l}_{\mathfrak{o}_{v_4}} \tilde{l}^{-1}}) \right) (\tilde{a}_t)$$

whereas

$$\frac{d}{dt} w_{\mathfrak{l}} w_{\tilde{l}^{-1}}(\tilde{a}_t) = \sum_{v \in \mathbb{V}_l} x_v \left(w_{\mathfrak{l}_{v,1}, \mathfrak{l}_{v,2}} w_{\tilde{l}^{-1}} + w_{\mathfrak{l}} w_{\tilde{l}_{v,3,1}^{-1}, \tilde{l}_{v,3,2}^{-1}} \right) (\tilde{a}_t).$$

By induction, for any regular loop of $\tilde{\mathbb{G}}_l$, with less than n intersections, $\text{tr}(H_\gamma)$ converges towards a constant. Therefore, bounding $|\text{tr}(H_p)|$ by 1 for any loop p , yields that, for any $t \in (0, 1)$,

$$\frac{d}{dt} \left(w_{\mathfrak{l}, \tilde{l}^{-1}}(\tilde{a}_t) - w_{\mathfrak{l}}(\tilde{a}_t)w_{\tilde{l}^{-1}}(\tilde{a}_t) \right) \rightarrow 0. \quad (8)$$

Combining (7) with (8) implies the result by dominated convergence. \square

We are now able to proof the Proposition 2.2.

⁷the definition is given above Lemma 3.

Proof of proposition 2.2. Let Φ , and Ψ be two functions satisfying the conditions of Proposition 2.2, with same function m . Then, according to 3.7 and 3.8, $\Phi(l) = \Psi(l)$, for all regular loops l of \mathbb{S}_T . The latter are dense in $L(\mathbb{S}_T)$ for the topology of given by convergence in length with fixed endpoints, but by assumption Φ and Ψ are continuous. \square

The last extension can be proven in general for any compact Riemann surfaces.

Proposition 3.9. *Suppose that $\text{tr}(H_l) \rightarrow \Phi_T(l)$ in probability as $N \rightarrow \infty$ for all regular loops l , then it does for all finite length loops of \mathbb{S}_T and Φ_T is continuous in length with fixed endpoints on $L(\mathbb{S}_T)$.*

This is a consequence of [32, Theorem 3.3.1], following the same argument given in [6, Theorem 4.1] for the whole plane case. For the sake of completeness, we recall these arguments in a slightly different form in the appendix.

4 Harmonic analysis in $U(N)$ and the discrete $\beta = 2$ Coulomb gas

4.1 A representation formula

Fix $T \in (0, \infty)$ and $N \in \mathbb{N}$. Let $(H_\gamma : \gamma \in P(\mathbb{S}_T))$ be a Yang–Mills holonomy field in $U(N)$ and let Λ_T^N be the random subset of $\mathbb{Z} + \frac{N-1}{2}$, whose distribution is given for $\lambda_1, \dots, \lambda_N \in N^{-1}(\mathbb{Z} + \frac{N-1}{2})$,

$$\mathbb{P}(\Lambda_T^N = \{\lambda_1, \dots, \lambda_N\}) = Z_T^{-1} \prod_{1 \leq i < j \leq N} (\lambda_i - \lambda_j)^2 \prod_{i=1}^N e^{-N\lambda_i^2 T/2} \quad (9)$$

with $Z_T = \sum_{\lambda_1 > \dots > \lambda_N, \lambda_i \in N^{-1}(\mathbb{Z} + \frac{N-1}{2})} \prod_{1 \leq i < j \leq N} (\lambda_i - \lambda_j)^2 \prod_{i=1}^N e^{-N\lambda_i^2 T/2}$, so that almost surely $\#\Lambda_T^N = N$. We shall consider the random probability measure on \mathbb{R} given by

$$\mu_T^N = \frac{1}{N} \sum_{x \in \Lambda_T^N} \delta_x.$$

Proposition 4.1. *Consider a simple loop $l \in L(\mathbb{S}_T)$ which divides the sphere into two components of area a and b . Fix $\delta > 0$ and denote by γ the positively oriented curve bounding the set $\{z \in \mathbb{C} : |z - x| < \delta \text{ for all } x \in I\}$, where I is the smallest interval containing the support of μ_T^N . Write $\bar{\gamma}$ for the curve with opposite orientation. Then, for any $n \in \mathbb{N}$,*

$$\begin{aligned} \mathbb{E}(\text{tr}(H_l^n)) &= \frac{e^{an^2/(2N)}}{2\pi i n} \mathbb{E} \int_{\bar{\gamma}} \exp\{n(az - G_{\mu_T^N}^{N/n}(z))\} dz \\ &= \frac{e^{-bn^2/(2N)}}{2\pi i n} \mathbb{E} \int_{\gamma} \exp\{-n(bz - G_{\mu_T^N}^{N/n}(z))\} dz \end{aligned}$$

where, for $\alpha \in \mathbb{R} \setminus \{0\}$ and $z \in \mathbb{C}$ with $d(z, \text{supp} \mu_T^N) > 1/|\alpha|$,

$$G_{\mu_T^N}^\alpha(z) = \alpha \int_{\mathbb{R}} \log \left(1 + \frac{1}{\alpha(z - x)} \right) \mu_T^N(dx).$$

Moreover, we have

$$\begin{aligned} \mathbb{E}(|\text{tr}(H_l^n)|^2) &= \frac{e^{(a+b)n^2/(2N)}}{(2\pi in)^2} \mathbb{E} \int_{\gamma} \exp\{n(az - G_{\mu_T}^{-N/n}(z))\} dz \\ &\quad \times \int_{\gamma} \exp\{n(bz - G_{\mu_T}^{-N/n}(z))\} dz. \end{aligned}$$

To prove these identities, we will use the decomposition of the heat kernel into characters of the compact group $U(N)$. The results we use may be found for example in [28]. The metric (1) on $U(N)$, regarded as an inner product on $\mathfrak{u}(N)$, induces an inner product on the dual space $\mathfrak{u}(N)^*$. Write \mathfrak{t}_N for the space of diagonal matrices with purely imaginary entries, and write \mathfrak{t}_N^* for its dual. Denote by $\|\cdot\|$ the norm on \mathfrak{t}_N^* induced by the inner product. Thus, if $e_k \in \mathfrak{t}_N$ is the diagonal matrix with k th entry equal to i and 0 elsewhere, we have for any $\mu \in \mathfrak{t}_N^*$,

$$\|\mu\|^2 = \frac{1}{N} \sum_{k=1}^N \mu(e_k)^2.$$

Denote by $(\omega^1, \dots, \omega^N)$ the basis dual to (e_1, \dots, e_N) and define $\rho = \sum_{i=1}^N \rho_i \omega^i \in \mathfrak{t}_N^*$ setting

$$2\rho = \sum_{1 \leq i < j \leq N} (\omega^i - \omega^j) = \sum_{i=1}^N (N+1-2i)\omega^i.$$

For $\lambda = (\lambda_1, \dots, \lambda_N) \in \mathbb{Z}^N$, define $\chi_\lambda : U(N) \rightarrow \mathbb{C}$ by

$$\chi_\lambda(g) = \frac{\det(e^{i\theta_p(\lambda_q + \rho_q)})_{1 \leq p, q \leq N}}{\det(e^{i\theta_p \rho_q})_{1 \leq p, q \leq N}} \quad (10)$$

where $\theta_1, \dots, \theta_N$ are the eigenvalues of g . The irreducible representations of $U(N)$ are then indexed by the set of non-increasing integer sequences λ and the adjunction-invariant eigenvectors of the Laplacian are given by the characters χ_λ , with

$$\Delta \chi_\lambda = (\|\rho\|^2 - \|\lambda + \rho\|^2) \chi_\lambda. \quad (11)$$

Write $L_{U(N)}^2(U(N))$ for the closed subspace of $L^2(U(N))$ consisting of functions which are invariant under adjunction by $U(N)$. Then the characters χ_λ form an orthonormal basis of $L_{U(N)}^2(U(N))$. In particular, we have

$$\int_{U(N)} \chi_\lambda(g) \chi_\mu(g^{-1}) dg = \delta_{\lambda, \mu}. \quad (12)$$

Hence we obtain the following formula for the heat kernel as an absolutely converging sum over characters: for any $g \in U(N)$ and all $T > 0$

$$q_T(g) = e^{\|\rho\|^2 T/2} \sum_{\lambda: \lambda_1 \geq \dots \geq \lambda_N} \chi_\lambda(1) \chi_\lambda(g) e^{-\|\lambda + \rho\|^2 T/2}. \quad (13)$$

The dimension of the associated representation is given by

$$\chi_\lambda(1) = \prod_{1 \leq i < j \leq N} \frac{\lambda_i + \rho_i - \lambda_j - \rho_j}{\rho_i - \rho_j}. \quad (14)$$

Denote by $V(\rho)$ the Vandermonde polynomial and rewrite the series for $q_T(1)$ as a sum over strictly increasing sequences to obtain

$$q_T(1) = \frac{e^{\|\rho\|^2 T/2}}{V(\rho)^2} \sum_{\mu \in (\mathbb{Z} + \frac{N-1}{2})^N : \mu_1 > \dots > \mu_N} \prod_{1 \leq i < j \leq N} (\mu_i - \mu_j)^2 e^{-\sum_{i=1}^N \mu_i^2 T/(2N)} \quad (15)$$

$$= \frac{e^{\|\rho\|^2 T/2}}{V(\rho)^2} \sum_{\substack{\mu \in N^{-1}(\mathbb{Z} + \frac{N-1}{2})^N \\ \mu_1 > \dots > \mu_N}} N^{N(N-1)} \prod_{1 \leq i < j \leq N} (\mu_i - \mu_j)^2 e^{-N \sum_{i=1}^N \mu_i^2 T/2} \quad (16)$$

The normalisation constant for the discrete β -ensemble (9) with $\beta = 2$, can now be rewritten as

$$\begin{aligned} Z_T &= V(\rho)^2 N^{-\frac{N(N-1)}{2}} \sum_{\lambda \in \mathbb{Z}^N : \lambda_1 \geq \dots \geq \lambda_N} \chi_\lambda(1)^2 e^{-\|\lambda + \rho\|^2 T/2} \\ &= V(\rho)^2 e^{-\|\rho\|^2 T/2} N^{-\frac{N(N-1)}{2}} q_T(1). \end{aligned}$$

Write \mathcal{O} for the subset of $(\mathbb{Z} + \frac{N-1}{2})^N$ given by tuples with all coordinates distinct. Thus

$$\mathcal{O} = (\mathbb{Z} + \frac{N-1}{2})^N \setminus \cup_{1 \leq i < j \leq N} \{x \in \mathbb{R}^N : x_i = x_j\}.$$

The set

$$\mathring{\Lambda} = \{n \in \mathcal{O} : n_1 > n_2 > \dots > n_N\} = \rho + \{\lambda \in \mathbb{Z}^N : \lambda_1 \geq \dots \geq \lambda_N\}$$

is a fundamental domain of \mathcal{O} for the action of the symmetric group: for any $\nu \in \mathcal{O}$, there is a unique $[\nu] \in \mathring{\Lambda}$ in the orbit of ν . Observe that if $x, y \in \mathcal{O}$, with $[x] = [y]$, then the function $\frac{\chi_{x-\rho}}{\chi_{y-\rho}}$ is constant and valued in $\{1, -1\}$, whereas for any $x, y \in \mathcal{O}$, the orthogonality relation (12) yields

$$\int_{U(N)} \chi_{x-\rho}(g) \chi_{y-\rho}(g^{-1}) dg = \frac{\chi_{x-\rho}(1)}{\chi_{y-\rho}(1)} \delta_{[x], [y]}. \quad (17)$$

To compute Wilson loops, we shall need to take the pointwise product of the trace on the fundamental representation \mathbb{C}^N with the characters. An elementary computation using (10) shows that⁸

$$\chi_\lambda(g) \text{Tr}(g^n) = \sum_{i \in \{1, \dots, N\} : \lambda + n\omega^i \in \mathcal{O}} \chi_{\lambda + n\omega^i}(g) \quad (18)$$

and

$$\chi_\lambda(g) \overline{\text{Tr}}(g^n) = \sum_{i \in \{1, \dots, N\} : \lambda - n\omega^i \in \mathcal{O}} \chi_{\lambda - n\omega^i}(g) \quad (19)$$

Proof of Proposition 4.1. From the definition of the Yang–Mills measure, we have

$$q_T(1) \mathbb{E}_{\text{YM}_T^N}(\text{tr}(H_t^n)) = \int_{U(N)} q_a(g) \text{tr}(g^n) q_b(g^{-1}) dg.$$

⁸More generally, such a formula holds true when considering in place of Tr the character of any irreducible representation with a minuscule highest weight (see [2]).

We expand the heat kernel in characters to obtain

$$\begin{aligned} & \int_{U(N)} q_a(g) \text{tr}(g^n) q_b(g^{-1}) dg \\ &= e^{\|\rho\|^2 T/2} \sum_{\lambda, \mu \in \mathring{\Lambda}} e^{-\|\lambda\|^2 a/2 - \|\mu\|^2 b/2} \chi_{\lambda-\rho}(1) \chi_{\mu-\rho}(1) \\ & \quad \times \int_{U(N)} \chi_{\lambda-\rho}(g) \text{tr}(g^n) \chi_{\mu-\rho}(g^{-1}) dg, \end{aligned}$$

The interchange of summation and integration here is valid because $a, b > 0$, which ensures absolute convergence. By orthogonality of characters (17) and the product rule (18), we have for any $\lambda, \mu \in \mathring{\Lambda}$,

$$\int_{U(N)} \chi_{\lambda-\rho}(g) \text{tr}(g^n) \chi_{\mu-\rho}(g^{-1}) dg = \frac{1}{N} \sum_{i: \lambda+n\omega_i \in \mathcal{O}} \frac{\chi_{\lambda+n\omega_i-\rho}(1)}{\chi_{\mu-\rho}(1)} \delta_{[\lambda+n\omega_i], \mu}$$

so

$$\begin{aligned} & \int_{U(N)} q_a(g) \text{tr}(g^n) q_b(g^{-1}) dg \\ &= e^{\|\rho\|^2 T/2} \sum_{\lambda \in \mathring{\Lambda}} \chi_{\lambda-\rho}(1)^2 e^{-\|\lambda\|^2 T/2} \frac{1}{N} \sum_{i: \lambda+n\omega_i \in \mathcal{O}} \frac{\chi_{\lambda+n\omega_i-\rho}(1)}{\chi_{\lambda-\rho}(1)} e^{(\|\lambda\|^2 - \|\lambda+n\omega_i\|^2)b/2}. \end{aligned} \tag{20}$$

We use the dimension formula (14) to rewrite the second sum, for $N > 1/\delta$, as follows

$$\begin{aligned} & \frac{1}{N} \sum_{i=1}^N \left(\prod_{j \neq i} \frac{\lambda_i + n - \lambda_j}{\lambda_i - \lambda_j} \right) e^{-nb\lambda_i/N - bn^2/(2N)} \\ &= \frac{e^{-bn^2/(2N)}}{2\pi i N n} \int_{N\gamma(\lambda)} \prod_{j=1}^N \left(1 + \frac{n}{z - \lambda_j} \right) e^{-nbz/N} dz \\ &= \frac{e^{-bn^2/(2N)}}{2\pi i n} \int_{\gamma(\lambda)} \exp \left\{ -nbz + \int_{\mathbb{R}} N \log \left(1 + \frac{n}{N(z-x)} \right) \mu_{\lambda/N}(dx) \right\} dz \end{aligned}$$

where $\gamma(\lambda)$ is a simple curve in the complex plane bounding $[\lambda_1/N, \lambda_N/N] + B(0, \delta)$ in an anti-clockwise way. Note that the product term in the left-hand-side vanishes when $\lambda + n\omega_i \notin \mathcal{O}$. On dividing by $q_T(1)$, we obtain

$$\begin{aligned} & \mathbb{E}_{\text{YM}_{T,N}}(\text{tr}(H_l^n)) \\ &= \frac{e^{-bn^2/(2N)}}{2\pi i n} \mathbb{E} \int_{\gamma_{T,N}(\delta)} \exp \left\{ -nbz + \int_{\mathbb{R}} N \log \left(1 + \frac{n}{N(z-x)} \right) \mu_{T,N}(dx) \right\} dz. \end{aligned}$$

The same argument using (19) in place of (18) yields the first formula. Then, using (18) for both heat kernels yields the third formula. \square

4.2 Concentration and hard edge properties for discrete β -ensembles

In order to prove Proposition 3.6, we need to understand the asymptotics of the discrete β -ensemble as $N \rightarrow \infty$. It has been proved in [15, 25] that the empirical measure converges towards a deterministic measure μ_T and satisfies a large deviation principle. In [34], this measure is computed explicitly as a minimizer of the large deviation rate function. In order to conclude, we need an estimate on the largest and smallest particles of the β -ensemble. For this purpose, we need to adapt slightly the arguments of [15, 25], which give a large deviation principle for the latter.

Proposition 4.2. *Let $l \in \mathcal{L}(\mathbb{S}_T)$ be a simple loop which divides \mathbb{S}_T into two domains, of area a and b . Then, under the Yang-Mills measure $\text{YM}_{N,T}$, for all $n \in \mathbb{N}$, as $N \rightarrow \infty$, we have*

$$\text{tr}(H_l^n) \rightarrow \Phi(l^n)$$

in probability, where

$$\Phi(l^n) = \frac{1}{2n\pi i} \int_{\gamma_T} \exp\{-n(bz - G_{\mu_T}(z))\} dz = \frac{1}{2n\pi i} \int_{\gamma_T} \exp\{n(az - G_{\mu_T}(z))\} dz$$

where γ_T is an anti-clockwise contour around the single cut of the Stieltjes transform

$$G_{\mu_T}(z) = \int_{\mathbb{R}} \frac{\mu_T(dx)}{z - x}$$

and where μ_T is the unique minimiser of the functional

$$\mathcal{I}_T(\mu) = \int_{\mathbb{R}^2} \left(\frac{(x^2 + y^2)T}{2} - 2 \log |x - y| \right) \mu(dx) \mu(dy)$$

over the set $\{\mu \in \mathcal{M}_1(\mathbb{R}) : \mu \leq \text{Leb}\}$. Moreover, setting $m_{T,N}(a) = \mathbb{E}[\text{tr}(H_l)]$ and $m_T(a) = \Phi_T(l)$, $m_{N,T} \rightarrow m_T$ uniformly in $C^1([0, T])$.

Let us note, that the above identities for the value of $\Phi(l)$ hold true specifically for the measure μ_T . Thus, for $b = 0$, as $G_{\mu_T}(z) \sim \frac{1}{z}$ as $z \rightarrow \infty$, the first integral is 1, so that indeed $\Phi(l) = 1$, but then the second equality is a priori non-trivial; a more symmetric formula is given in Lemma 4.4.

Corollary 4.3. *Under the hypothesis of Corollary 4.2, in the case $T \in (0, \pi^2]$, we have*

$$\Phi_T(l^n) = \int_{\mathbb{R}} e^{inx} s_{\sqrt{ab/T}}(dx)$$

where

$$s_t(dx) = \frac{\sqrt{(4t - x^2)^+} dx}{2\pi t}.$$

Proof. It is well known that the minimum of \mathcal{I}_T over $\mathcal{M}_1(\mathbb{R})$ is achieved at the semicircle law

$$s_{1/T}(dx) = \frac{\sqrt{(4T - (Tx)^2)^+} dx}{2\pi}.$$

For $T \leq \pi^2$, we have $s_T \in \mathcal{M}_c$ so $\mu_T = s_T$. Then

$$G_{\mu_T}(z) = \frac{Tz - \sqrt{(zT)^2 - 4T}}{2}$$

is a one-to-one mapping from $\mathbb{C} \setminus [-2\sqrt{T}, 2\sqrt{T}]$ to the unit disc of radius $\frac{1}{\sqrt{T}}$, reversing the orientation and with inverse mapping $z + \frac{T}{z}$. As $G_{\mu_T}(z) = G_{\mu_1}(\frac{z}{T})$, to show the above formula, it is enough to consider $T = 1$. The following was first observed in [4, 9]. Changing variable with $z \mapsto iz$ leads to

$$\begin{aligned} \int_{\gamma_T} e^{n(az - G_{\mu_1}(z))} \frac{dz}{2\pi i} &= \frac{1}{i} \int_C e^{ni(\frac{a}{y} + by)} (1 + \frac{1}{y^2}) \frac{dy}{2\pi i} = n(a+b) \sum_{k \geq 0} \frac{(-nab)^k}{k!(k+1)!} \\ &= n \int_{\mathbb{R}} e^{inx} s_{\sqrt{ab}}(dx). \end{aligned}$$

Then, using Proposition 4.2 and a scaling argument, the claimed formula holds for all $T \leq \pi^2$. \square

When $T > \pi^2$, the expression for the minimizer μ_T is more involved, see [34], where the following is obtained (therein formula (4.12))

$$G_{\mu_T}(z) = \frac{zT}{2} - \frac{2}{\beta z} \sqrt{(z^2 - \alpha^2)(z^2 - \beta^2)} \int_0^1 \frac{ds}{(1 - \alpha^2 s^2 / z^2) \sqrt{(1 - s^2)(1 - k^2 s^2)}} \quad (21)$$

where $k = \alpha/\beta \in (0, 1)$ and

$$\text{supp}(\mu_T) = [-\beta, \beta], \quad \text{supp}(\text{Leb} - \mu_T) = \mathbb{R} \setminus (-\alpha, \alpha).$$

It can be shown that μ_T has a Hölder density ρ_T with respect to Lebesgue measure with $\rho_T(x) = 1$ for $|x| \leq \alpha$ while, for $|x| \in (\alpha, \beta)$,

$$\rho_T(x) = \frac{2\sqrt{(x^2 - \alpha^2)(\beta^2 - x^2)}}{\pi\beta|x|} \int_0^1 \frac{ds}{(1 - \alpha^2 s^2 / x^2) \sqrt{(1 - s^2)(1 - k^2 s^2)}}. \quad (22)$$

Moreover, for $|x| \in [\alpha, \beta]$, in the limit $z \rightarrow x$ with $z \notin \mathbb{R}$, we have

$$\Re(G(z)) \rightarrow \text{PV} \int \frac{\rho_T(y)dy}{x - y} = \frac{xT}{2}. \quad (23)$$

We can now prove the second equality appearing in Proposition 4.2 using the following formula.

Lemma 4.4. *For all $a, b \geq 0$ with $a + b = T$, and all $n \in \mathbb{N}$, we have*

$$\begin{aligned} \frac{1}{2\pi i n} \int_{\gamma_T} \exp\{-n(bz - G_{\mu_T}(z))\} dz &= \frac{2}{n\pi} \int_{\alpha}^{\beta} \cosh\{(a - b)nx/2\} \sin\{n\pi\rho_T(x)\} dx \\ &= \frac{1}{2\pi i n} \int_{\gamma_T} \exp\{n(az - G_{\mu_T}(z))\} dz \end{aligned}$$

where γ_T is an anti-clockwise contour around $\text{supp}(\mu_T)$.

Proof. Since the integrand of the left-hand side is holomorphic in $\mathbb{C} \setminus [-\beta, \beta]$, we can assume that γ_T is the anti-clockwise boundary of $[-\beta - \varepsilon, \beta + \varepsilon] \times [-\varepsilon, \varepsilon]$ for any $\varepsilon > 0$. Now, as ρ_T is Hölder, according to Plemelj-Sokhotski formula [19], G_{μ_T} can be continuously extended as G_+ and G_- , on $\overline{\mathbb{H}} = \{z \in \mathbb{C} : \Im(z) \geq 0\}$ and $-\overline{\mathbb{H}}$, with

$$G_{\pm}(x) = \text{PV} \int_{\mathbb{R}} \frac{\rho_T(y)}{x - y} \mp i\pi\rho_T(x)$$

for any $x \in \mathbb{R}$. Using the dominated convergence theorem on $\overline{\mathbb{H}}$ and $-\overline{\mathbb{H}}$, it follows that the left-hand-side is equal to

$$\int_{-\beta}^{\beta} e^{-n(bx - \text{PV} \int \frac{\rho_T(y)dy}{x-y})} \sin(n\pi\rho_T(x)) \frac{dx}{n\pi}.$$

When $x \in [-\alpha, \alpha]$, since $\rho_T(x) = 1$, the integrand vanishes. As ρ_T is symmetric, according to (23), the first equality follows. The same argument yields the second equality as well. \square

The next lemma is a variation on the large deviation upper bound in Theorem 2.2 of [25].

Lemma 4.5. *For all $a, T > 0$, there are constants $C, R < \infty$ such that, for all $N \in \mathbb{N}$,*

$$\mathbb{E} \left(e^{a\Lambda_{T,1}^N} 1_{\{\Lambda_{T,1}^N > R\}} \right) \leq C e^{-CN}.$$

Proof. For $N, M \in \mathbb{N}$, set

$$Z_{N,M} = \sum_{\substack{\lambda \in M^{-1}\mathbb{Z}^N \\ \lambda_1 < \dots < \lambda_N}} \prod_{1 \leq i < j \leq N} (\lambda_i - \lambda_j)^2 \prod_{i=1}^N e^{-M\lambda_i^2 T/2}.$$

and consider the random set $\tilde{\Lambda}_T^N$ of \mathbb{Z} with N elements, such that for all $\lambda_1, \dots, \lambda_N \in \frac{1}{N}\mathbb{Z}$,

$$\mathbb{P}(\tilde{\Lambda}_T^N = \{\lambda_1, \dots, \lambda_N\}) = Z_{N,N}^{-1} \prod_{1 \leq i < j \leq N} (\lambda_i - \lambda_j)^2 \prod_{i=1}^N e^{-N\lambda_i^2 T/2}$$

and denote by $\tilde{\mu}_{T,N}$ the associated empirical measure. Then, for all $a, R > 0$,

$$\mathbb{E} \left(e^{aN\Lambda_{T,1}^N} 1_{\{\Lambda_{T,1}^N > R\}} \right) \leq e^{1/4} \frac{\mathbb{E} \left(e^{(a+1)\tilde{\Lambda}_{T,1}^N} 1_{\{\tilde{\Lambda}_{T,1}^N > R - \frac{1}{2}\}} \right)}{\mathbb{E} \left(e^{-\tilde{\Lambda}_{T,1}^N} \right)}.$$

Besides, for all $a > 0$,

$$\begin{aligned} & \mathbb{E} \left(e^{a\tilde{\Lambda}_{N,T}} 1_{\{\tilde{\Lambda}_{N,T} > R\}} \right) \\ &= \frac{Z_{N-1,N}}{Z_{N,N}} \sum_{s \in N^{-1}\mathbb{Z}} e^{as - Ns^2 T/2} 1_{\{s > R\}} \mathbb{E} \left(\exp \left\{ 2N \int \log |s - x| \tilde{\mu}_{T,N-1}(dx) \right\} \right), \end{aligned}$$

By [25, Lemma 4.1], as $N \rightarrow \infty$, we have

$$\frac{1}{N} \log \mathbb{E} \left(\exp \left\{ N \int \log(1 + x^2) \tilde{\mu}_{T,N-1}(dx) \right\} \right) \rightarrow c_T = \int \log(1 + x^2) \tilde{\mu}_T(dx).$$

We use the bound $|s - x|^2 \leq (1 + s^2)(1 + x^2)$ to see that there exists $c > 2c_T$ such that, for all N ,

$$\mathbb{E} \left(e^{a\Lambda_{T,1}^N} 1_{\{\Lambda_{T,1}^N > R\}} \right) \leq c e^{cN} \frac{Z_{N-1,N}}{Z_{N,N}} \sum_{s \in N^{-1}\mathbb{Z}} e^{as - Ns^2 T/2} (1 + s^2)^N 1_{\{s > R\}}.$$

Now, by [25, Lemma 4.5],

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \log \frac{Z_{N-1,N}}{Z_{N,N}} < \infty.$$

The claim follows by choosing R suitably large. \square

Proof of Proposition 4.2. Let us first prove that $\mathbb{E}(\text{tr}(H_l^n)) \rightarrow \Phi(l^n)$. For any $\lambda \in \mathring{\Lambda}$, evaluating (18) at 1 yields $N\chi_{\lambda-\rho}(1) = \sum_{i=1}^N \chi_{\lambda-\rho+n\omega_i}(1)$, so that

$$\frac{1}{N} \sum_{i=1}^N \frac{\chi_{\lambda-\rho+n\omega_i}(1)}{\chi_{\lambda-\rho}(1)} e^{b(\|\lambda\|^2 - \|\lambda+\omega_i\|^2)} \leq e^{a\lambda_1/N}.$$

Let us identify the random set Λ_T^N with a non-increasing sequence of $\mathbb{Z} + \frac{N-1}{2}$. The expression (20) can be reformulated as

$$\mathbb{E}_{\text{YM}_{T,N}}[\text{tr}(H_l)] = \mathbb{E}\left[\frac{1}{N} \sum_{i=1}^N \frac{\chi_{\Lambda_T-\rho+n\omega_i}(1)}{\chi_{\Lambda_T-\rho}(1)} e^{\frac{b}{2}(\|\Lambda_T\|^2 - \|\Lambda_T+\omega_i\|^2)}\right].$$

Choosing $R > 0$ as in Lemma 4.5, it follows from the latter that

$$\mathbb{E}_{\text{YM}_{T,N}}[\text{tr}(H_l)] = \mathbb{E}[1_{\lambda_{T,1} \leq R} \frac{1}{N} \sum_{i=1}^N \frac{\chi_{\Lambda_T-\rho+n\omega_i}(1)}{\chi_{\Lambda_T-\rho}(1)} e^{\frac{b}{2}(\|\Lambda_T\|^2 - \|\Lambda_T+\omega_i\|^2)}] + \varepsilon_N. \quad (24)$$

with $\varepsilon_N \rightarrow 0$ as $N \rightarrow \infty$. Let C_{R+1} be the circle of radius $R+1 > 0$, endowed with an anti-clockwise orientation. If $\Lambda_{T,1}, -\Lambda_{T,N} \leq R$ then, according to Lemma 4.1,

$$\frac{1}{N} \sum_{i=1}^N \frac{\chi_{\Lambda_T-\rho+n\omega_i}(1)}{\chi_{\Lambda_T-\rho}(1)} e^{\frac{b}{2}(\|\Lambda_T\|^2 - \|\Lambda_T+\omega_i\|^2)} = \frac{e^{-\frac{bn^2}{2N}}}{n} \int_{C_{R+1}} \exp(-n(bz - G_{\mu_{N,T}}^{N/n}(z))) \frac{dz}{2i\pi}. \quad (25)$$

According to Theorem 3.2 in [15], $(\mu_{N,T})_{N \geq 1}$ satisfies a large deviation principle with good rate function \mathcal{I}_T on \mathcal{M}_c . In particular, $\mu_{T,N}$ converges in probability towards μ_T . Moreover, when $(\mu_N)_{N \geq 1}$ is a sequence of $\mathcal{M}_1([-R, R])$, that converges weakly towards $\mu \in \mathcal{M}_1([-R, R])$, $G_{\mu_N}^{N/n}(z)$ converges uniformly over $\mathbb{C} \setminus B(R+1)$ towards $G_\mu(z) = \int \frac{\mu(dx)}{z-x}$. As $\mathbb{P}(\max\{|\Lambda_{T,1}|, |\Lambda_{T,N}|\} \leq NR) \rightarrow 1$ as $N \rightarrow \infty$, it follows from (24) and (25) that

$$\mathbb{E}_{\text{YM}_{T,N}}[\text{tr}(H_l^n)] \rightarrow \frac{1}{n} \int_{C_{R+1}} e^{-n(bz - G_{\mu_T}(z))} \frac{dz}{2i\pi} = \Phi(l^n).$$

To show that $\text{tr}(H_l^n) \rightarrow \Phi(l^n)$ in probability, it is enough to prove that the covariance $\text{cov}(\text{tr}(H_l^n), \overline{\text{tr}(H_l^n)})$ tends to 0 as $N \rightarrow \infty$. According to Lemma 4.1, $\text{cov}[\overline{\text{tr}(H_l^n)}, \text{tr}(H_l^n)]$ equals

$$\frac{e^{\frac{a+b}{2N}}}{n^2} \text{cov}\left(\left(\int_{\gamma_{T,N}^{\leftarrow}} \exp(n(az - G_{\mu_T}^{-N/n}(z))) \frac{dz}{2i\pi}\right), \left(\int_{\gamma_{T,N}^{\leftarrow}} \exp(-n(bz - G_{\mu_T}^{-N/n}(z))) \frac{dz}{2i\pi}\right)\right).$$

The same argument as above shows that this second covariance vanishes as $N \rightarrow \infty$. \square

5 Some properties of the master field

5.1 Relation with linear Hermitian Bridges

Let $(s_t)_{t \geq 0}$ be a free Brownian motion in a non-commutative probability space $(\mathcal{A}, 1, *, \phi)$. If $(K_t)_{t \geq 0}$ is a Brownian motion on $(\mathfrak{u}_N, \langle \cdot, \cdot \rangle)$, then, $(K_t)_{t \geq 0}$ converges in non-commutative distribution towards $(s_t)_{t \geq 0}$. It follows that the Brownian bridge from 0 to 0, $(K_{t,T})_{0 \leq t \leq T} = (K_t - \frac{t}{T}K_T)_{0 \leq t \leq T}$ converges as well in non-commutative distribution towards $(s_t - \frac{t}{T}s_T)_{0 \leq t \leq T}$. This process is called the free Brownian bridge and we shall denote it by $(s_{t,T})_{0 \leq t \leq T}$. In the subcritical regime, where $T < \pi^2$, we can now see that the empirical measure of the time marginal of a Brownian bridge of length T on $U(N)$ behaves as the one of the exponential mapping of a Brownian bridge of the same length on $\mathfrak{u}(N)$. However, their full trajectories viewed as a non-commutative process differ.

Proposition 5.1. *For $T \in (0, \pi^2)$, for all $t \in [0, T]$ and all $n \in \mathbb{Z}^*$,*

$$\tau(b_{t,T}^n) = \frac{1}{2\pi i n} \int_{\gamma} e^{n(tz - G_{\mu_T}(z))} dz = \tau(s_{t,T}^n).$$

On the other hand, the non-commutative distributions of $(b_{t,T})_{0 \leq t \leq T}$ and $(e^{is_{t,T}})_{0 \leq t \leq T}$ differ: for almost all $0 < r < t < T \leq \pi^2$, $\tau(b_{r,T}^{-1}b_{t,T}) \neq \tau(e^{-is_{r,T}}e^{is_{t,T}})$.

Proof. The first assertion is the content of Corollary 4.3. Let $(\ell_t)_{0 \leq t \leq T}$ be a family of simple loops as in section 2.2. For any $r, t \in (0, T)$ with $r < t$, $\ell_t \ell_r^{-1}$ is a the limit in $L(\mathbb{S}_T)$ of simple loops all bounding a domain of area $t - r$. Hence $H_{\ell_t \ell_r^{-1}} = H_{\ell_r}^{-1} H_{\ell_t}$ has same law as $H_{\ell_{t-r}}$. Therefore, a unitary Brownian Bridge satisfies that $B_{r,T}^{-1} B_{t,T}$ has same law as $B_{t-r,T}$. In particular, for $t \leq \pi^2$, $\tau(b_{r,T}^{-1}b_{t,T}) = \tau(b_{t-r,T}) = \tau(e^{is_{t-r,T}})$. Besides, by free independence and stationarity of the increments of the free Brownian motion, $s_{t-r,T}$ has the same non-commutative distribution as $s_{t,T} - s_{r,T}$. Hence, $\tau(b_{r,T}^{-1}b_{t,T}) = \tau(e^{i(s_{t,T} - s_{r,T})})$. We shall show yet that for almost all $0 < r < t < T$, $\tau(e^{-is_{r,T}}e^{is_{t,T}}) \neq \tau(e^{i(s_{t,T} - s_{r,T})})$. Let us set $F(x, y, T) = \tau(e^{-is_{x,T}}e^{is_{y,T}} - e^{i(s_{y,T} - s_{x,T})})$, for $x, y \in (0, 1)$, with $x < y$. The function $G_{x,y}(s) = F(x^2, y^2, s^2)$ is analytic in the variable s on \mathbb{C} and it is enough to conclude to show that one of its derivative is non-zero at 0, for all $0 < x < y < 1$. What is more, expanding the exponential function up to order 4 and using scale invariance of the free Brownian motion leads to¹⁰ $G_{x,y}^{(4)}(0) = 2\tau(s_{x,1}^2 s_{y,1}^2 - s_{x,1} s_{y,1} s_{x,1} s_{y,1})$. The variables $(s_{x,1})_{0 \leq x \leq 1}$ are semi-circular, therefore all free cumulants in them of order larger than 3 vanish (see for instance (11.4) in [37]). Using the decomposition of moments into free cumulants¹¹ (see (11.8) of [37]),

$$\tau(s_{x,1}^2 s_{y,1}^2 - s_{x,1} s_{y,1} s_{x,1} s_{y,1}) = \tau(s_{x,1}^2) \tau(s_{y,1}^2) - \tau(s_{x,1} s_{y,1})^2 = x(y - x)(1 - y) > 0.$$

□

⁹this statement follows also from Markov property and invariance by multiplication of the Brownian motion.

¹⁰observe nonetheless that $G_{x,y}^{(k)}(0) = 0$, for $k \leq 3$.

¹¹here it can be understood as a "non-commutative" Wick formula, with non-crossing matchings in place of all matchings.

5.2 Duality in the middle of the bridge

For any time $T > 0$, the spectral measure of $b_{T/2,T}$ can be related to the measure μ_T in the following way. This relation appeared first in the physics literature in [22, formula (1.2)], without a mathematical proof.

Proposition 5.2. *For any $T > 0$, the spectral measure of $b_{T,\frac{T}{2}}$ has a density ρ_T^* with respect to the Lebesgue measure $d\theta$ on \mathbb{U} (of mass 2π), invariant under complex conjugation, such that*

$$\pi\rho_T^* : \mathbb{U} \cap \mathbb{H} \rightarrow (\alpha, \beta)$$

is the inverse mapping of

$$e^{i\pi\rho_T} : (\alpha, \beta) \rightarrow \mathbb{U} \cap \mathbb{H}.$$

Proof. Let us first note that the function $\rho_T : (\alpha, \beta) \rightarrow (0, 1)$ is strictly decreasing. Indeed, according to formula (22) and an elementary computation (see for example formula (150) of [35]), for $x \in (\alpha, \beta)$,

$$\frac{\pi\alpha}{2} \sqrt{(x^2 - \alpha^2)(\beta^2 - x^2)} \rho_T'(x) = \int_0^1 \frac{\alpha^2 s^2 - x^2}{\beta^2 \sqrt{(1-s^2)(1-k^2 s^2)}} ds < 0.$$

As ρ_T is continuous on \mathbb{R} , with $\rho_T(\alpha) = 1$ and $\rho_T(\beta) = 0$, $\pi\rho_T : (\alpha, \beta) \rightarrow (0, \pi)$ is a one-to-one mapping. Let us denote by $\psi : (0, \pi) \rightarrow (\alpha, \beta)$ its inverse. Let us consider $T = a + b$. According, to Lemma (4.4),

$$\begin{aligned} \tau(b_{\frac{T}{2},T}^n) &= \int_{\alpha}^{\beta} \sin(n\pi\rho_T(x)) \frac{2dx}{n\pi} \\ &= \int_0^{\pi} \sin(n\theta) \frac{-2\psi'(\theta)d\theta}{n\pi}. \end{aligned}$$

An integration by part leads then to

$$\tau(b_{\frac{T}{2},T}^n) = 2 \int_0^{\pi} \cos(n\theta) \psi(\theta) \frac{d\theta}{\pi}.$$

It follows that the spectral measure of $b_{\frac{T}{2},T}$ has a density with respect the Lebesgue measure on \mathbb{U} that is invariant by complex conjugation and equal to

$$\rho_T^*(\theta) = \frac{\psi(\theta)}{\pi},$$

for any $\theta \in (0, \pi)$. □

5.3 Convergence towards the planar master field

We shall here investigate the behavior of the master field Φ_T as $T \rightarrow \infty$.

Proposition 5.3. *For any $t > 0$ and $n \in \mathbb{N}^*$, as $T \rightarrow \infty$,*

$$\tau(b_{t,T}^n) \rightarrow \frac{e^{-nt/2}}{n} \int_{\gamma} \left(1 + \frac{1}{z}\right)^n e^{-ntz} \frac{dz}{2i\pi} = \frac{e^{-nt/2}}{n} \sum_{k=1}^{n-1} \frac{(-nt)^k}{k!} \binom{n}{k+1},$$

where γ is any counterclockwise oriented simple contour the complex planen winding once around 0.

Proof. Let us show that $\alpha, \beta \rightarrow \frac{1}{2}$, as $T \rightarrow \infty$. Since $\mu_T \in \mathcal{M}_1$, if $\alpha_{T_n} \rightarrow \alpha_\infty < \frac{1}{2}$, for some sequence $(T_n)_{n \geq 0}$, then $\beta_{T_n} \not\rightarrow \frac{1}{2}$. Up to a subsequence extraction, one can assume that $\alpha \rightarrow \alpha_\infty < \frac{1}{2}$ and $\beta \rightarrow \beta_\infty \in (\frac{1}{2}, \infty)$. For any probability measure $\mu = \rho(|x|)dx \in \mathcal{M}_c$, with $\rho(x) = 1$, for $x \leq \alpha$ and $\rho(x) = 0$, for $x \geq \beta$, $\mathcal{I}_T(\mu_T) - \mathcal{I}_T(\mu)$ is equal to

$$\int_{\mathbb{R}^2} \log|x-y|(\mu_T(dx)\mu_T(dy) - \mu(dx)\mu(dy)) + 2T \int_\alpha^\beta x^2(\rho_T(x) - \rho(x))dx.$$

By definition of μ_T , $\mathcal{I}_T(\mu_T) \leq \mathcal{I}_T(\mu)$. Since ρ_T is decreasing and $\liminf \int_\alpha^\beta \rho_T(x)dx \geq \frac{1}{2} - \alpha_\infty$, there exist $x_0 \in (\alpha_\infty, 1/2)$, ρ , such that the function g defined by $g(t) = \rho_T(x_0 + t) - \rho(x_0 + t) > 0$, and $g(-t) = -g(t)$ for $0 \leq t < \min(x_0 - \alpha_\infty, \beta_\infty - x_0)$ does not depend on T . For this choice $\mathcal{I}_T(\mu_T) - \mathcal{I}_T(\mu) \rightarrow +\infty$. The claim follows by contradiction. Therefore, μ_T converges weakly towards $1_{[-\frac{1}{2}, \frac{1}{2}]}$ as $T \rightarrow \infty$, and uniformly on any compact of $\{z \in \mathbb{C} : |z| > \frac{1}{2}\}$,

$$G_{\mu_T}(z) \rightarrow \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{dx}{z-x} = \log \left(\frac{z + \frac{1}{2}}{z - \frac{1}{2}} \right),$$

for any determination log of the complex logarithm with a cut in the closure of the lower half plane. For any $T > t$, according to 4.2, when C is a counterclockwise circle of radius $R > \frac{1}{2}$, as $T \rightarrow \infty$,

$$\begin{aligned} \tau(b_{t,T}) &= \frac{1}{n} \int_C e^{-ntz+nG_{\mu_T}(z)} \frac{dz}{2i\pi} \rightarrow \frac{1}{n} \int_C e^{-ntz} \left(\frac{z + \frac{1}{2}}{z - \frac{1}{2}} \right)^n \frac{dz}{2i\pi} \\ &= \frac{e^{-\frac{nt}{2}}}{n} \int_{C-\frac{1}{2}} e^{-nty} \left(1 + \frac{1}{y} \right)^n \frac{dy}{2i\pi}. \end{aligned}$$

□

Let x_∞ be the point with highest third coordinate of the Euclidean sphere \mathbb{S}_T of area T , in \mathbb{R}^3 , centered at 0. Let $L(\mathbb{R}^2)$ be the set of loops of finite length in \mathbb{R}^2 and $\Phi : L(\mathbb{R}^2) \rightarrow [-1, 1]$, the planar master field as it is defined in [32].

Proposition 5.4. *For any family $(\psi_T)_{T>0}$ of area of preserving diffeomorphisms $\psi_T : \mathbb{C} \rightarrow \mathbb{S}_T \setminus \{x_\infty\}$, and any loop fixed $l \in L(\mathbb{R}^2)$, as $T \rightarrow \infty$,*

$$\Phi_T(\psi_T(l)) \rightarrow \Phi(l).$$

Let us recall that the field Φ satisfies as well Makeenko-Migdal equations as in Theorem 3. One can show using an argument very close to the one used in proposition 3.8, where the sphere is replaced with the plane, that it is sufficient to prove the statement for powers of simple loops of \mathbb{R}^2 . Then, Proposition 5.3 implies the statement. For the sake of conciseness, we shall not reproduce the proof here.

6 Combinatorial formulas for the master field

In the physics article [38], Rusakov proposed without proof, that it is possible to get a closed formula for the value of the master field for any regular loop on the sphere.

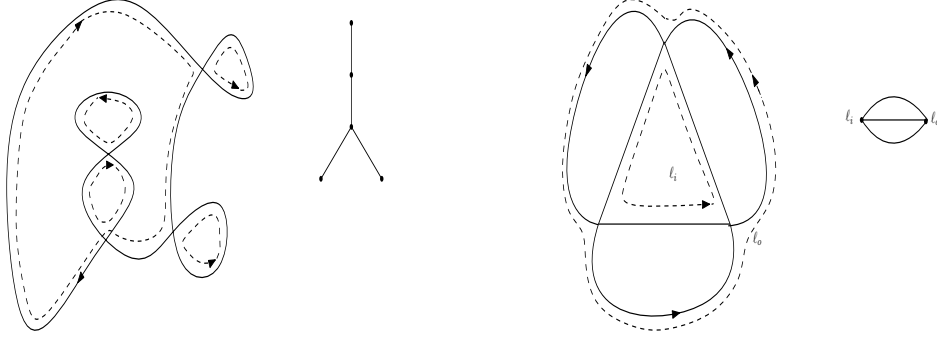


Figure 4: Two combinatorial loops with their graph $\mathcal{G}_{\mathfrak{l}}$ drawn aside, together with $\mathcal{C}_{\mathfrak{l}}$ drawn in dashed lines. The left-one is planar, whereas the right one is not.

In what follows, we shall prove formulas with a slightly different form than the one given in [38] and for a restricted class of loops, introduced in [26], when studying the planar master field. We will also need the following property.

Proposition 6.1. *For any combinatorial loop \mathfrak{l} and any fixed T , the function*

$$\begin{aligned} \Delta_{\mathfrak{l}}(T) &\longrightarrow [-1, 1] \\ a &\longmapsto \Phi(l), \text{ with } l \in L(\mathbb{G}_{\mathfrak{l}}, a) \end{aligned}$$

admits an analytic extension to $\Delta_{\mathfrak{l}}(T)^{\mathbb{C}} = \{z \in \mathbb{C}^{\mathbb{F}} : \sum_{F \in \mathbb{F}} z_F = T\}$, that we denote by $\Phi_{\mathfrak{l}}$.

Proof. When \mathfrak{l} is a maximally winding loop around the faces F_0, F_{∞} , and $l \in L(\mathbb{G}_{\mathfrak{l}}, a)$ is smoothly parametrized, with a supported on $\{F_0, F_{\infty}\}$; then l can be reduced to a loop s^n , with $n \in \mathbb{Z}$ and $s \in L(\mathbb{S}_T)$ is simple; according to Corollary 4.2, if $n \neq 0$,

$$\Phi(l) = \frac{1}{n} \int_{\gamma_T} e^{n(a(F_{\infty})z - G_{\mu_T}(z))} \frac{dz}{2\pi i}.$$

The general case follows by induction on the number of intersections of \mathfrak{l} using the Makeenko–Migdal equations along the line given by (6), as in the proof of Proposition 2.2. \square

A combinatorial loop \mathfrak{l} is called *planar* if for any of its intersection point $v \in \mathbb{V}_{\mathfrak{l}}$, $\mathfrak{l}_{v,1}$ and $\mathfrak{l}_{v,2}$ do not intersect each other. We denote their set by \mathfrak{L}_0 . For any $v \in \mathbb{V}_{\mathfrak{l}}$, we shall label the loop among $\mathfrak{l}_{v,1}, \mathfrak{l}_{v,2}$ using the left outgoing edge out of v , by $\mathfrak{l}_{v,l}$ (and resp. $\mathfrak{l}_{v,r}$ for the right outgoing edge). For any combinatorial loop \mathfrak{l} , there is a collection \mathcal{C} of simple combinatorial loops in $\mathbb{G}_{\mathfrak{l}}$, such that each edge of $\mathbb{G}_{\mathfrak{l}}$ is used exactly once by some element of \mathcal{C} and with the same orientation as \mathfrak{l} . We denote their set by $\mathcal{S}_{\mathfrak{l}}$. We set $\mathcal{C}_{\mathfrak{l}} \in \mathcal{S}_{\mathfrak{l}}$ the unique collection such that its elements do not cross transversally one another. We endow $\mathcal{C}_{\mathfrak{l}}$ with a graph structure $\mathcal{G}_{\mathfrak{l}}$, setting that two simple loops \mathfrak{s} and $\tilde{\mathfrak{s}} \in \mathcal{C}_{\mathfrak{l}}$ are neighbor if they intersect themselves (see Figure 4). The following lemma is elementary and left to the Reader.

Lemma 6.2. *For any $\mathcal{C} \in \mathcal{S}_{\mathfrak{l}}$,*

$$g_{\mathfrak{l}} := \#\mathbb{V}_{\mathfrak{l}} + 1 - \#\mathcal{C} \in 2\mathbb{N}$$

does not depend on \mathcal{S}_l or on the choice of \mathcal{C} . The following conditions are equivalent

1. l is planar
2. $\mathcal{S}_l = \{\mathcal{C}_l\}$.
3. $g_l = 0$.
4. \mathcal{G}_l is a tree.

For any combinatorial loop l and $\mathcal{C} \in \mathcal{S}_l$, the collection \mathcal{C} is entirely characterized by the subset $\mathbb{V}_{\mathcal{C}}$ of vertices of \mathbb{G}_l , which are non-crossing points of the loops of \mathcal{C} . For any $\mathcal{C} \in \mathcal{S}_l$,

$$\#\mathbb{V}_{\mathcal{C}} = \#\mathcal{C} - 1 = \#\mathbb{V}_l - g_l.$$

Then, for any $v \in \mathbb{V}_l$, $l_{v,l}$ and $l_{v,r}$ are also planar, and the pair $\mathcal{C}_{l_{v,l}}$ and $\mathcal{C}_{l_{v,r}}$ is a partition of \mathcal{C}_l . Let us denote by $\mathring{\mathcal{C}}_l$, the set of vertices of \mathcal{G}_l with degree bigger than 2. For any $T > 0$, any planar combinatorial loop l , $F_{\infty} \in \mathbb{F}_l$, and $\mathcal{C} \in \mathcal{S}_l$, we say that a family of closed simple contours $(\gamma_{\mathfrak{s}})_{\mathfrak{s} \in \mathcal{C}}$ of \mathbb{C} is T -admissible if

1. the curves $(\gamma_{\mathfrak{s}})_{\mathfrak{s} \in \mathcal{C}}$ do not intersect each others.
2. for any $\mathfrak{s} \in \mathcal{C}$, $\gamma_{\mathfrak{s}}$ is a simple loop around $[-\beta, \beta]$, with same orientation as the one of \mathfrak{s} around F_{∞} .
3. if $\gamma_{\mathfrak{s}_2}$ and $\gamma_{\mathfrak{s}_1}$ are two neighboring contours, then \mathfrak{s}_1 and \mathfrak{s}_2 are two neighbors in \mathcal{G}_l .
4. if \mathfrak{s}_o is a leaf of \mathcal{G}_l with neighbor \mathfrak{s}_i , then the bounded connected component of $\mathbb{C} \setminus \gamma_{\mathfrak{s}_o}$ includes $\gamma_{\mathfrak{s}_i}$.

For any loop l in \mathbb{G}_l and $F \in \mathbb{F}_l$, recall that¹² $n_{F,l} \in \mathbb{Z}^{\mathbb{F}_l}$ denotes the winding number function of l , with $n_{F,l}(F) = 0$. For any $\mathcal{C} \in \mathcal{S}_l$, $\mathfrak{s} \in \mathcal{C}$ and $F \in \mathbb{F}_l$, we write $\varepsilon_{\mathfrak{s}} \in \{-1, 1\}$ for the non-zero value of $n_{F,\mathfrak{s}}$. When $v \in \mathbb{V}_{\mathcal{C}}$, we further denote by $\mathfrak{s}_{v,l}$ and $\mathfrak{s}_{v,r}$ the two loops of \mathcal{C} going through v , respectively on the left and on the right. We set

$$\mathcal{O}_{\mathcal{C}} = \mathbb{C}^{\mathcal{C}} \setminus (\cup_{\mathfrak{s} \neq \mathfrak{r} \in \mathcal{C}} \{z : z_{\mathfrak{s}} = z_{\mathfrak{r}}\}).$$

Proposition 6.3. *Let l be a planar combinatorial loop, $F_{\infty} \in \mathbb{F}_l$ and $T > 0$. Then, for any $a \in \Delta_l(T)$ and any T -admissible contours $(\gamma_{\mathfrak{s}})_{\mathfrak{s} \in \mathcal{C}_l}$,*

$$\Phi_l(a) = (2i\pi)^{-\#\mathbb{V}_l-1} \int_{\gamma_{\mathfrak{s}_0}} dz_{\mathfrak{s}_0} \dots \int_{\gamma_{\mathfrak{s}_m}} dz_{\mathfrak{s}_m} Q_{l,F_{\infty}}(a, z),$$

for any ordering $(\mathfrak{s}_i)_{i=0}^m$ of \mathcal{C}_l , where for any $z \in \mathcal{O}_{\mathcal{C}}$,

$$Q_{l,F_{\infty}}(a, z) = \frac{e^{\sum_{\mathfrak{s} \in \mathcal{C}_l} (\varepsilon_{\mathfrak{s}} G_{\mu_T}(z_{\mathfrak{s}}) + z_{\mathfrak{s}} \langle n_{F_{\infty}, \mathfrak{s}}, a \rangle)}}{\prod_{v \in \mathbb{V}_l} (z_{\mathfrak{s}_{v,r}} - z_{\mathfrak{s}_{v,l}})}.$$

Proof of Proposition 6.3. The contours $(\gamma_{\mathfrak{s}})_{\mathfrak{s} \in \mathcal{C}_l}$ being non-crossing, the function $Q_{l,F}(a, z)$ is continuous, analytic in a , and uniformly bounded when $(z_{\mathfrak{s}})_{\mathfrak{s} \in \mathcal{C}_l}$ is parametrizing $(\gamma_{\mathfrak{s}})_{\mathfrak{s} \in \mathcal{C}_l}$ and a belongs to a compact subset of $\Delta_l(T)^{\mathbb{C}}$. Therefore, the right-hand-side is a well defined multiple contour integral, that does not¹³ depend on the choice of order of integration, and has an analytic extension to $\Delta_l(T)^{\mathbb{C}}$. For

¹²cf. section 3.3

¹³but it depends on the relative positions of the contours in the complex plane.

any planar combinatorial loop \mathfrak{l} and $a \in \Delta_{\mathfrak{l}}(T)$, let $\Psi_{\mathfrak{l}}(a)$ denote the difference of the right-hand-side $\varphi_{\mathfrak{l}}(a)$ with the left-hand-side $\Phi_{\mathfrak{l}}(a)$. According to proposition 6.1, $\Psi_{\mathfrak{l}}$ is an analytic function on $\Delta_{\mathfrak{l}}(T)^{\mathbb{C}}$. We shall prove by induction on the number of intersection points of \mathfrak{l} , that $\varphi_{\mathfrak{l}}$ does not depend on the choice of T -admissible contour and that $\Psi_{\mathfrak{l}} = 0$. When \mathfrak{l} is simple, the statement follows from Proposition 4.2. Let us prove the induction step and therefor consider a planar loop \mathfrak{l} with n intersections. For any $v \in \mathbb{V}_{\mathfrak{l}}$, let $F_{\infty, \mathfrak{l}} \in \mathbb{F}_{\mathfrak{l}_{v, \mathfrak{l}}}$ and $F_{\infty, r} \in \mathbb{F}_{\mathfrak{l}_{v, r}}$ be the unique faces of resp. $\mathbb{G}_{\mathfrak{l}_{v, \mathfrak{l}}}$ and $\mathbb{G}_{\mathfrak{l}_{v, r}}$ containing F_{∞} , and for any $a \in \Delta_{\mathfrak{l}}(T)$, let $a_{\mathfrak{l}} \in \Delta_{\mathfrak{l}_{v, \mathfrak{l}}}(T)^{\mathbb{C}}$ and $a_r \in \Delta_{\mathfrak{l}_{v, r}}(T)^{\mathbb{C}}$ denote the complex extension of the affine maps of area, $\iota_{\mathbb{G}_{\mathfrak{l}_{v, \mathfrak{l}}}}^{\mathbb{G}_{\mathfrak{l}}}(a)$ and $\iota_{\mathbb{G}_{\mathfrak{l}_{v, r}}}^{\mathbb{G}_{\mathfrak{l}}}(a)$. Then, for any $a \in \Delta_{\mathfrak{l}}(T)^{\mathbb{C}}$ and $\mathfrak{s} \in \mathcal{C}$, $\mu_v \langle n_{F_{\infty, \mathfrak{s}}}, a \rangle$ takes the value 1, if \mathfrak{s} uses the right outgoing edge at v , -1 if it uses the left one, and 0, if it does not go through v . Therefore, for any $z \in \mathcal{O}_{\mathcal{C}}$,

$$\mu_v Q_{\mathfrak{l}, F_{\infty}}(a, z) = Q_{\mathfrak{l}_{v, \mathfrak{l}}, F_{\infty, \mathfrak{l}}}(a_{\mathfrak{l}}, z_{\mathfrak{s}}, \mathfrak{s} \in \mathcal{C}_{\mathfrak{l}_{v, \mathfrak{l}}}) Q_{\mathfrak{l}_{v, r}, F_{\infty, r}}(a_r, z_{\mathfrak{s}}, \mathfrak{s} \in \mathcal{C}_{\mathfrak{l}_{v, r}}). \quad (26)$$

We shall now use this equation for a suitable choice of $v \in \mathbb{V}_{\mathfrak{l}}$.

Since $n \geq 1$, $\mathcal{G}_{\mathfrak{l}}$ has at least two leaves, and one of the them is not the boundary of the face F_{∞} . Denote such a leaf by \mathfrak{s}_0 , \mathfrak{s}_i its neighbor in $\mathcal{G}_{\mathfrak{l}}$ and by F_c the component of its complement which does not include F_{∞} . Then $\mathcal{G}_{\mathfrak{l}_{v, \mathfrak{l}}}$ is the tree obtained from $\mathcal{G}_{\mathfrak{l}}$ by removing the edge $(\mathfrak{s}_0, \mathfrak{s}_i)$ and the vertex \mathfrak{s}_0 , whereas $\mathcal{G}_{\mathfrak{l}_{v, r}} = \{\mathfrak{s}_0\}$, or vice-versa with the left indices replaced by the right ones. It implies that the contours $(\gamma_{\mathfrak{s}})_{\mathfrak{s} \in \mathcal{C}_{\mathfrak{l}_{v, \mathfrak{l}}}}$ and $(\gamma_{\mathfrak{s}})_{\mathfrak{s} \in \mathcal{C}_{\mathfrak{l}_{v, r}}}$ are T -admissible. Since the right-hand-side of (26) is uniformly bounded on any compact subset of $\Delta_{\mathfrak{l}}(T)^{\mathbb{C}} \times \mathcal{O}_{\mathcal{C}_{\mathfrak{l}}}$, we deduce that

$$\mu_v \varphi_{\mathfrak{l}}(a) = \varphi_{\mathfrak{l}_{v, \mathfrak{l}}}(a_{\mathfrak{l}}) \varphi_{\mathfrak{l}_{v, r}}(a_r),$$

for any $a \in \Delta_{\mathfrak{l}}(T)^{\mathbb{C}}$. Besides, using the Makeenko-Migdal equations of Theorem 3.1, and Proposition 6.1, $\mu_v \Phi_{\mathfrak{l}}(a) = \Phi_{\mathfrak{l}_{v, \mathfrak{l}}}(a_{\mathfrak{l}}) \Phi_{\mathfrak{l}_{v, r}}(a_r)$, for any $a \in \Delta_{\mathfrak{l}}(T)$. As $\mathfrak{l}_{v, \mathfrak{l}}$ and $\mathfrak{l}_{v, r}$ are planar with less than $n - 1$ intersection points, it follows that for any $a \in \Delta_{\mathfrak{l}}(T)^{\mathbb{C}}$,

$$\mu_v \Psi_{\mathfrak{l}}(a) = 0. \quad (27)$$

We check now the boundary condition of this equation. The loop \mathfrak{l} being planar, there exists a planar combinatorial loop $\tilde{\mathfrak{l}}$, with exactly $n - 1$ intersections, an affine map $\iota_c : \Delta_{\mathfrak{l}}(T) \cap \{a : a(F_c) = 0\} \rightarrow \Delta_{\tilde{\mathfrak{l}}}(T)$ and a distinguished face $\tilde{F}_{\infty} \in \mathbb{F}_{\tilde{\mathfrak{l}}}$, such that for any $a \in \Delta_{\mathfrak{l}}(T)$, with $a(F_c) = 0$, $\mathfrak{l}(a) \cap \tilde{\mathfrak{l}}(\iota_c(a)) \neq \emptyset$, and $\iota_c(a)(\tilde{F}_{\infty}) = 0$ if and only if $a(F_{\infty}) = 0$. By continuity of the master field Φ_T , for the topology in 1-variation on $L(\mathbb{S}_T)$, such equivalence classes of loops satisfy for all $a \in \Delta_{\mathfrak{l}}(T)$,

$$\Phi_{\mathfrak{l}}(a) = \Phi_{\tilde{\mathfrak{l}}}(\iota_c(a)), \quad (28)$$

Furthermore, by analyticity of $\Phi_{\mathfrak{l}}$ and $\Phi_{\tilde{\mathfrak{l}}}$, this equality holds true for all $a \in \Delta_{\mathfrak{l}}(T)^{\mathbb{C}}$ with $a(F_c) = 0$. Let $\nu \in \mathbb{Z}^{\mathbb{F}_{\mathfrak{l}}}$ be the vector with $\nu(F_c) = 1$, that is proportional to μ_v , viewed as an element of $\{1_{\mathbb{F}_{\mathfrak{l}}}\}^{\perp} \cap \mathbb{R}^{\mathbb{F}_{\mathfrak{l}}}$, where v is the only vertex neighboring F_c . Then, thanks to (27), for all $a \in \Delta_{\mathfrak{l}}(T)^{\mathbb{C}}$,

$$\Psi_{\mathfrak{l}}(a) = \Psi_{\mathfrak{l}}(a - a(F_c)\nu).$$

As $a - a(F_c)\nu \in \Delta_{\mathfrak{l}}(T)^{\mathbb{C}} \cap \{a : a(F_c) = 0\}$, thanks to (28), in order to conclude, it is sufficient to show that for all $a \in \Delta_{\mathfrak{l}}(T)^{\mathbb{C}}$ with $a(F_c)$,

$$\varphi_{\mathfrak{l}}(a) = \varphi_{\tilde{\mathfrak{l}}}(\iota_c(a)).$$

Now, for such a vector a , for all $z \in \mathcal{O}_{\mathcal{C}_l}$,

$$Q_{l,F_\infty}(a, z) = Q_{\tilde{l}, \tilde{F}_\infty}(a, (z_{\mathfrak{s}})_{\mathfrak{s} \in \mathcal{C}_l \setminus \{s_0\}}) \frac{\varepsilon_{s_0} e^{\varepsilon_{s_0} G_{\mu_T}(z_{s_0})}}{z_{s_0} - z_{s_i}}.$$

For any $a \in \Delta_l(T)^\mathbb{C}$, the only singularity of $z_{s_0} \in \mathbb{C} \setminus [-\beta, \beta] \mapsto Q_{l,F_\infty}(a, z)$ is at z_{s_i} . Hence, the set of contours being T -admissible, one can assume that the unbounded component of $\mathbb{C} \setminus \gamma_{s_0}$ does not intersect any $\gamma_{s'}$ with $s' \neq s$. Therefore, for any $(z_{\mathfrak{s}})_{\mathfrak{s} \neq s_0}$ parametrizing $(\gamma_{\mathfrak{s}})_{\mathfrak{s} \neq s_0}$,

$$\int_{\gamma_{s_0}} \frac{dz_{s_0}}{2i\pi} Q_{l,F_\infty}(a, z) = Q_{\tilde{l}, \tilde{F}_\infty}(a, (z_{\mathfrak{s}})_{\mathfrak{s} \in \mathcal{C}_l \setminus \{s_0\}}) \int_C \frac{dz_{s_0}}{2i\pi} \frac{e^{\varepsilon_{s_0} G_{\mu_T}(z_{s_0})}}{z_{s_0} - z_{s_i}},$$

with C a counterclockwise circle with center 0, whose interior contains all contours $(\gamma_{\mathfrak{s}})_{\mathfrak{s} \neq s_0}$. As $G_{\mu_T}(z) \sim \frac{1}{z}$, when $z \rightarrow \infty$, it follows that

$$\int_C \frac{dz_{s_0}}{2i\pi} \frac{e^{\varepsilon_{s_0} G_{\mu_T}(z_{s_0})}}{z_{s_0} - z_{s_i}} = - \int_{1/C} \frac{dy}{2i\pi} \frac{e^{\varepsilon_{s_0} G_{\mu_T}(\frac{1}{y})}}{y(1 - y z_{s_i})} = 1.$$

Therefore, performing the integration in $\varphi_l(a)$, first with respect to s_0 , one gets that, when $a \in \Delta_l(T)^\mathbb{C}$, with $a(F_c) = 0$,

$$\begin{aligned} \varphi_l(a) &= (2i\pi)^{-\#\mathbb{V}_l} \int_{\gamma_{s_1}} dz_{s_1} \dots \int_{\gamma_{s_m}} dz_{s_m} Q_{\tilde{l}, \tilde{F}_\infty}(a, (z_{\mathfrak{s}})_{\mathfrak{s} \in \mathcal{C}_l \setminus \{s_0\}}) \\ &= \varphi_l(\iota_c(a)). \end{aligned}$$

□

A From regular loops to rectifiable loops

For the ease of the Reader, we collect and reformulate in this appendix results and observations of [32, Theorem 3.3.1] and [6, Theorem 4.1]; the arguments of [6] are explained therein in the plane but work similarly on any compact Riemann surface as we shall see. The only purpose of the reformulation is to emphasize that the argument only relies on the analysis for the rectifiable loops and not on the structure group in which the holonomy take values.

Let Σ be a compact Riemann surface, $x \in \Sigma$ a fixed point, and denote by $A(\Sigma)$ the space of piecewise geodesic loops based at x . Two loops $\alpha, \beta \in A(\Sigma)$ are said equivalent and we write $\alpha \sim \beta$ if they can be obtained from one another by multiple insertion or deletion of backtracking geodesic paths. A pseudo-distance on $A(\Sigma)$ is a symmetric function $d : A(\Sigma) \times A(\Sigma) \rightarrow \mathbb{R}_+$ such that for all $\alpha, \beta, \gamma \in A(\Sigma)$,

$$d(\alpha, \beta) \leq d(\alpha, \gamma) + d(\gamma, \beta)$$

and

$$d(\alpha, \alpha) = 0.$$

We say it is *homogeneous* if for any $\alpha, \beta, \gamma \in A(\Sigma)$,

- $d(\alpha, \beta) \leq d(\alpha, \gamma) + d(\gamma, \beta)$

- $d(\alpha^{-1}, \beta^{-1}) = d(\alpha, \beta)$
- $d(\gamma\alpha, \gamma\beta) = d(\alpha, \beta)$
- $d(\alpha', \beta') = d(\alpha, \beta)$, for any $\alpha' \sim \alpha, \beta' \sim \beta$.

For any $K > 0$, we say it is K -regular if for any simple loop $s \in A(\Sigma)$, of length smaller than K^{-1} , bounding a domain of area a ,

$$d(1, s) \leq K\sqrt{a}.$$

The arguments of section 3.3 of [32] apply verbatim to prove the following statement.

Theorem A.1. *Assume that $d : A(\Sigma) \times A(\Sigma) \rightarrow \mathbb{R}_+$ is a K -regular homogeneous pseudo-distance on $A(\Sigma)$, for some $K > 0$, then it admits a unique continuous extension to $L(\Sigma) \times L(\Sigma)$ for the topology of convergence in length with fixed endpoints.*

It can be also seen as a consequence¹⁴ of [32, Theorem 3.3.1].

Proof. Let (Γ, d) be a completion of $(A(\Sigma), d)$, that is, a metric space (Γ, d) and a map $\psi : A(\Sigma) \rightarrow \Gamma$, such that $d(\alpha, \beta) = d(\psi(\alpha), \psi(\beta))$, for any $\alpha, \beta \in A(\Sigma)$. As d is homogeneous, it is elementary to check that Γ is a group, with identity element the constant loop at x , with continuous operations for d and that $\psi : A(\Sigma) \rightarrow \Gamma$ is a multiplicative map for concatenation on $A(\Sigma)$. As d is K -regular and homogeneous, it follows that Γ satisfies the condition of [32, Theorem 3.3.1]. The latter implies that $\hat{\psi} : A(\Sigma) \rightarrow \Gamma, \gamma \mapsto \gamma^{-1}$ admits a continuous extension Ψ to $L(\Sigma)$. The map $\mathbf{d} : L(\Sigma) \times L(\Sigma) \rightarrow \mathbb{R}_+, (\alpha, \beta) \mapsto d(\Psi(\alpha), \Psi(\beta))$ is then a continuous extension of d . \square

Let $\tilde{A}(\Sigma)$ be the subset of $A(\Sigma)$ of piecewise geodesic loops which have only simple intersections.

Porism A.2. *Assume that $K > 0$ and $(d_n)_{n \geq 0}$ is a sequence of K -regular homogeneous pseudo-distances converging pointwise on $\tilde{A}(\Sigma)$ towards $d : \tilde{A}(\Sigma) \times \tilde{A}(\Sigma) \rightarrow \mathbb{R}_+$. Then, d admits a unique continuous extension to $L(\Sigma)$ and the extensions of $(d_n)_{n \geq 0}$ to $L(\Sigma)$ converge pointwise to the one of d .*

Proof of the Porism. We shall use here some arguments of the proof of [32, Theorem 3.3.1] for each distance $d_p, p \geq 0$, applied to the multiplicative function $\hat{\psi}_p$ valued in the completion (Γ_p, d_p) of $A(\Sigma)$, as considered in the proof of Theorem A.1. Let $l \in L(\Sigma)$ be a loop parametrized by $\tilde{l} : [0, 1] \rightarrow \Sigma$, with speed almost everywhere constant. For any $m \in \mathbb{N}^*$, such that $2^{-m}\ell(l)$ is strictly smaller than the cut-locus κ of Σ , we set $D_m(l)$ the smallest piecewise geodesic loop, going consecutively through $\tilde{l}(0), x_1, \dots, x_{2^m-1}$, where for all $k \in \{1, \dots, 2^m-1\}$, $\text{dist}_\Sigma(x_k, \tilde{l}(k2^{-m})) < 4^{-m}$, so that $D_m(l) \in \tilde{A}(\Sigma)$. For any $l \in L(\Sigma)$, let us denote its length by $\mathcal{L}(l)$. Then, according to the second inequality of the proof¹⁵ of [32, Prop. 3.3.9], for any $n \geq m \geq \kappa$

$$\begin{aligned} d_p(D_n(l), D_m(l)) &= d_p(\hat{\psi}_p(D_m(l)), \hat{\psi}_p(D_n(l))) \\ &\leq K(\mathcal{L}(l) + 2^{-m+1})^{3/4}(\mathcal{L}(D_n(l)) - \mathcal{L}(D_m(l)) + 2^{-m+1})^{1/4}. \end{aligned}$$

¹⁴It can be proven to be equivalent to it. We prove here only one implication.

¹⁵Note that there is a typo there in [32], the term $\ell(c)$ should be replaced by $\ell(D_m(c))$.

The inequality implies that the extension of d_p satisfies for any $l \in L(\Sigma)$, $m, p \in \mathbb{N}$, with $2^m \kappa > \mathcal{L}(l)$,

$$d_p(l, D_m(l)) \leq K(\mathcal{L}(l) + 2^{-m+1})^{3/4}(\mathcal{L}(l) - \mathcal{L}(D_m(l)) + 2^{-m+1})^{1/4}. \quad (29)$$

Therefore, the pointwise convergence of the extensions of $(d_p)_{p \geq 0}$ to $L(\Sigma)$ follows and their limit is the unique continuous extension of d . \square

Proof of Prop. 3.9. For any $N > 1, \alpha, \beta \in A(\Sigma)$, let us set

$$d_N(\alpha, \beta) = \text{tr}((H_\alpha - H_\beta)(H_\alpha - H_\beta)^*)^{1/2} = \sqrt{2(1 - \Re(W_{\alpha^{-1}\beta}))}.$$

Almost surely under $\text{YM}_{N,T}$, $(H_l)_{l \in L(\Sigma)}$ is multiplicative and therefore d_N is a homogeneous pseudo-distance on $A(\Sigma)$. Setting for all $\alpha, \beta \in L(\Sigma)$,

$$\bar{d}_N(\alpha, \beta) = \mathbb{E}[d_N(\alpha, \beta)^2]^{1/2} = \sqrt{2(1 - \mathbb{E}[W_{\alpha^{-1}\beta}])}^{1/2}$$

defines an homogeneous pseudo-distance on $A(\Sigma)$. With the notation of Proposition 4.2, for any simple loop l , bounding an area $a \in [0, T]$, $\bar{d}_N(1, l) = \sqrt{2(1 - m_{T,N}(a))}^{1/2} \leq C \min(a, T - a)^{1/2}$, with $C = \sup_{N \geq 1} \|m'_{N,T}\|_\infty < \infty$, as $m_{T,N}(0) = 1 = m_{T,N}(1)$. Hence, \bar{d} is C -regular. Since W_α converges in probability towards $\Phi_T(\alpha)$ for all $\alpha \in A(\Sigma)$, it follows that setting for all $\alpha, \beta \in A(\Sigma)$, $d(\alpha, \beta) = \sqrt{2(1 - \Phi(\alpha^{-1}\beta))}^{1/2}$ defines an homogeneous pseudo-distance. We can apply the porism A.2, to get that for any $\alpha, \beta \in L(\Sigma)$, $\bar{d}_N(\alpha, \beta) \rightarrow d(\alpha, \beta)$. The map $l \in L(\Sigma) \mapsto \frac{2-d(1,l)^2}{2}$ is a continuous extension of Φ and for any $l \in L(\Sigma)$,

$$\mathbb{E}[\text{tr}(H_l)] \rightarrow \Phi(l),$$

as $N \rightarrow \infty$. Note that for any $\alpha, \beta \in L(\Sigma)$, using Cauchy-Schwarz inequality twice,

$$\mathbb{E}[|\text{tr}(H_\alpha) - \text{tr}(H_\beta)|] \leq \mathbb{E}(\text{tr}((H_\alpha - H_\beta)(H_\alpha - H_\beta)^*)^{1/2}) \leq \bar{d}_N(\alpha, \beta).$$

Since $\text{tr}(H_\alpha)$ converges in probability towards $\Phi(\alpha)$, for any $\alpha \in A(\Sigma)$, the inequality (29) applied to \bar{d}_N , shows that for any $l \in L(\Sigma)$, $\mathbb{E}[|\text{tr}(H_l) - \Phi(l)|]$ vanishes as $N \rightarrow \infty$. \square

Acknowledgments. The first author wishes to thank Guillaume Cébron, Franck Gabriel, Thierry Lévy, and Mylène Maïda for several motivating and fruitful discussions about this project.

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